has no rest frame, so what does (the frame-dependent) eqn. $\varepsilon^\mu = (0, \hat{\varepsilon})$ mean? I.e., what frame does it refer to? The answer is that it's the frame in which Coulomb gauge is valid. Coulomb gauge is just a gauge convention that you can use in any Lorentz frame you wish, but once it's set up, if you go to another frame (transforming A^μ as a 4-vector) you will no longer have Coulomb gauge. Coulomb gauge is implied by our use of the Hamiltonian

$$H_1 = e \int d^3\vec{x} \quad \bar{\psi}(\vec{x})\, \vec{\gamma}\cdot\vec{A}(\vec{x})\, \psi(\vec{x}) \,.$$

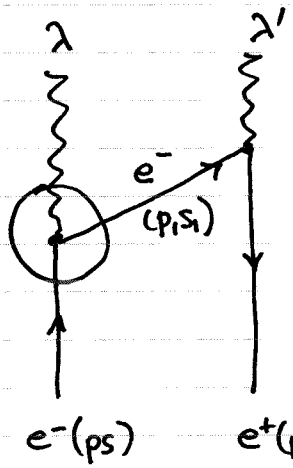
Anyway, with these conventions,

$$\gamma^\mu \varepsilon^*_\mu = \not{\varepsilon}^* = -\vec{\gamma}\cdot\hat{\varepsilon}^* \,.$$

Next, we do the spatial integral in $\langle k|H_1|i\rangle$, which absorbs one factor of $1/V$ and gives a δ-fn. in momentum. The result is

$$\langle k|H_1|i\rangle = -e\sqrt{\frac{2\pi}{V}}\sqrt{\frac{m^2}{EE_1\omega}}\,(\bar{u}_{p_1 s_1}\, \not{\varepsilon}^*\, u_{ps})\; \delta_{\vec{p}_1,\, \vec{p}-\vec{k}}$$

The momentum δ-fn. implies momentum conservation at the 1st vertex of diagram 1a,



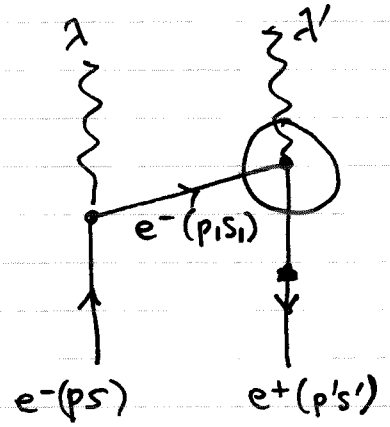
That is,

$$\vec{p} = \vec{p}_1 + \vec{k} \,.$$

Similarly we compute the matrix element $\langle n|H_1|k\rangle$, finding

$$\langle n|H_1|k\rangle = -e\sqrt{\frac{2\pi}{V}}\sqrt{\frac{m^2}{E'E_1\omega'}}\left(\bar{v}_{p's'}\,\not{\epsilon}'^{*}\,u_{p_1s_1}\right)\delta_{\vec{p}_1,\vec{k}'-\vec{p}'}\,.$$

Now the Kron. δ represents momentum conservation at the 2nd vertex of diagram 1a,



That is,

$$\vec{p}' + \vec{p}_1 = \vec{k}'$$

Now we put it together, getting

$$M_{1a} = e^2\,\frac{2\pi}{V}\sqrt{\frac{m^2}{EE'\omega\omega'}}\sum_{p_1s_1}\frac{m}{E_1}\;\frac{\left(\bar{v}_{p's'}\,\not{\epsilon}'^{*}\,u_{p_1s_1}\right)\left(\bar{u}_{p_1s_1}\,\not{\epsilon}^{*}\,u_{ps}\right)}{E-\omega-E_1}$$

$$\times\;\delta_{\vec{p}_1,\vec{p}-\vec{k}}\;\delta_{\vec{p}_1,\vec{k}'-\vec{p}'}$$

The momentum sum p_1 can be done because of the Kron. δ's; after we do this $\vec{p}_1$ is understood to be $\vec{p}_1 = \vec{p}-\vec{k} = \vec{k}'-\vec{p}'$, and $E_1 = \sqrt{m^2+|\vec{p}_1|^2}$, $p_1^\mu = (E_1, \vec{p}_1)$. The spin sum s_1 can be done using the energy projection operators,

$$\Lambda_+(p_1) = \sum_{s_1} u_{p_1s_1}\,\bar{u}_{p_1s_1} = \frac{\not{p}_1 + m}{2m}\,.$$

Altogether, this gives...

$$M_{1a} = e^2 \frac{2\pi}{V} \sqrt{\frac{m^2}{EE'\omega\omega'}} \frac{1}{2E_1} \frac{\bar{v}_{p's'} \not{\epsilon}'^* (\not{p}_1 + m) \not{\epsilon}^* u_{ps}}{E - \omega - E_1} \delta_{\vec{p}+\vec{p}', \vec{k}+\vec{k}'}$$

where

$$\vec{p}_1 = \vec{p} - \vec{k} = \vec{k}' - \vec{p}'$$

$$E_1 = \sqrt{m^2 + |\vec{p}_1|^2}$$

$$p_1^\mu = (E_1, \vec{p}_1)$$

Similarly we treat M_{2a}. There are two main things that are different. Since we are creating and destroying an intermediate position state, we get

$$\sum_{s_2} v_{p_2 s_2} \bar{v}_{p_2 s_2} = \frac{\not{p}_2 - m}{2m} = -\Lambda_-(p_2)$$

when summing over intermediate states.

Second, in the matrix element $\langle k | H_1 | i \rangle$, we have the sequence of operators $d_{p's'} a^\dagger_{k'} d^\dagger_{p_2 s_2}$ (see the table), which are not <u>normal ordered</u>. When we swap $d_{p's'} d^\dagger_{p_2 s_2}$ we get a <u>minus sign</u> due to the Fermion nature of the electron. This is vital, since we are adding <u>amplitudes</u>, not probabilities. In summary, we find

crucial minus sign ↓

$$M_{2a} = -e^2 \frac{2\pi}{V} \sqrt{\frac{m^2}{EE'\omega\omega'}} \frac{1}{2E_2} \frac{\bar{v}_{p's'} \not{\epsilon}'^* (\not{p}_2 - m) \not{\epsilon}^* u_{ps}}{E' - E_2 - \omega'} \delta_{\vec{p}+\vec{p}', \vec{k}+\vec{k}'}$$

where $\vec{p}_2 = \vec{p}' - \vec{k}' = \vec{k} - \vec{p}$ $\qquad p_2^\mu = (E_2, \vec{p}_2)$.

$$E_2 = \sqrt{m_e^2 + |\vec{p}_2|^2}$$

The Kronecker $\delta_{\vec{p}_0+\vec{p}',\,\vec{k}+\vec{k}'}$ means that momentum is conserved overall in the Feynman diagram. This is a common factor to all 4 Feynman diagrams.

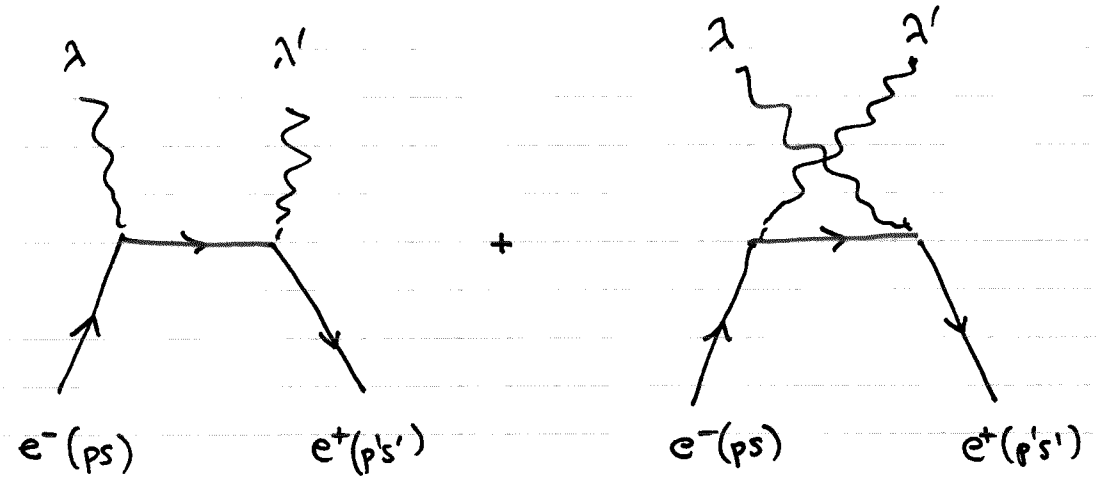
The amplitudes for cases 1b, 2b are obtained from those for 1a, 2a just by swapping $\lambda \leftrightarrow \lambda'$, that is, $\hat{\varepsilon} \leftrightarrow \hat{\varepsilon}'$, $\omega \leftrightarrow \omega'$, $\vec{k} \leftrightarrow \vec{k}'$.

The amplitudes M_{1a}, M_{2a} have many factors in common, so performing the sum boils down to adding the parts that are different,

$$\frac{1}{2E_1}\,\frac{\not{p}_1+m}{E-\omega-E_1} - \frac{1}{2E_2}\,\frac{\not{p}_2-m}{E'-E_2-\omega'}\,.$$

The algebra of doing this is straightforward. What is interesting is that the result can be written in covariant form.

We are using "old fashioned perturbation theory." It is based on Hamiltonians and the selection of a privileged reference frame (which is always implied by the use of a Hamiltonian.) It is not wrong, but it is not as elegant or efficient as covariant perturbation theory. In covariant perturbation theory, diagrams 1a and 2a are combined into a single diagram, and likewise 1b and 2b. The two diagrams are



$M = \quad M_a \quad + \quad M_b .$

although we are using old fashioned perturbation theory, if our answers are correct they can be put into covariant form. Let us combine the fractions above to do that. We try to make the fraction 2 look like fraction 1. First note that $\vec{p}_2 = -\vec{p}_1$, so $E_2 = E_1$ and $p_2^\mu = (E_1, -\vec{p}_1)$. Thus,

$$\not{p}_1 + \not{p}_2 = 2E_1 \gamma^0,$$

$$\not{p}_1 - \not{p}_2 = -2\vec{p}_1 \cdot \vec{\gamma}.$$

Next, by conservation of energy, $E + E' = \omega + \omega'$, the denominator in the 2nd term can be written,

$$E' - E_2 - \omega' = -(E - \omega) - E_1 .$$

So,

$$\text{fractions} = \frac{1}{2E_1}\left[\frac{\not{p}_1 + m}{(E-\omega) - E_1} + \frac{\not{p}_2 - m}{(E-\omega) + E_1}\right]$$

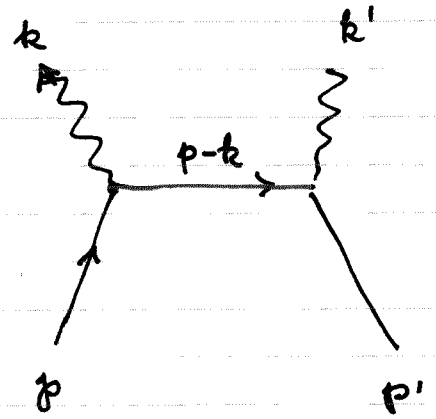
$$= \frac{1}{2E_1} \frac{(E-\omega+E_1)(\not{p}_1+m) + (E-\omega-E_1)(\not{p}_2+m)}{(E-\omega)^2 - E_1^2} .$$

The denominator can be written

$$(E-\omega)^2 - |\vec{p}_1|^2 - m^2 = (E-\omega)^2 - (\vec{p}-\vec{k})^2 - m^2$$

$$= (p^\mu - k^\mu)(p_\mu - k_\mu) - m^2 = (p-k)^2 - m^2$$

where $(p-k)^2$ means the Lorentz scalar product. Notice that $p-k$ is the 4-momentum along the internal line, if we demand conservation of 4-momenta at each vertex:



Notice that old fashioned pert. theory gives conservation of 3-momenta at each vertex, not 4-momenta.

As for the numerator it is

$$(E-\omega)(\not{p}_1+\not{p}_2) + E_1(\not{p}_1-\not{p}_2) + 2mE_1$$

$$= (E-\omega)\,2E_1\,\gamma^0 \cdot \gamma^0 - 2E_1\,\vec{\gamma}\cdot\vec{p}_1 + 2mE_1$$

$$= 2E_1\left[(E-\omega)\gamma^0 - \vec{\gamma}\cdot(\vec{p}-\vec{k}) + m\right] = 2E_1(\not{p}-\not{k}+m).$$

The fractions can thus be ~~cancelled~~ combined into

$$\frac{\not p - \not k + m}{(p-k)^2 - m^2} = D_F(p-k),$$

where D_F is the Feynman propagator,

$$D_F(p) = \frac{\not p + m}{p^2 - m^2}.$$

Thus,

$$M_a = e^2 \frac{2\pi}{V} \sqrt{\frac{m^2}{EE'\omega\omega'}} \left(\bar{u}_{p's'} \not\epsilon'^* D_F(p-k) \not\epsilon^* u_{ps}\right) \delta_{\vec{p}+\vec{p}', \vec{k}+\vec{k}'}$$

The 4-momentum $p-k$ associated with the internal line is generally not <u>on mass shell</u>, that is, it does not satisfy the energy-momentum relation $(p-k)^2 = m^2$. In fact if it did, the denominator in the Feynman propagator would diverge. This divergence is just like the ones we have seen in 2nd order perturbation theory (with a continuum of states, as in the Lamb shift), and it is regularized with the $i\varepsilon$ prescription. The Feynman propagator is really a Green's function (in momentum space) for the Dirac equation.

Notice that the difference between our diagrams 1a and 2a

was the time sequencing of the creation and destruction of the intermediate state. One way gives an electron, the other a positron, in the intermediate state. But if the 4-momentum p-k is space-like, then the time ordering of its two ends depends on the Lorentz frame, and you can't tell (i.e., there is no Lorentz invariant meaning) whether it is an electron or positron.

The remaining steps to combine $M_a + M_b = M$, to square it, sum and average over spins if you don't care about them, to do the Dirac spin operations (traces of γ matrices) etc. to get the cross-section are straightforward, but there is some algebra involved. The calculation is summarized in Sakurai, where the final cross-section is quoted.