Some notes on the <u>Gordon decomposition</u> of the current in Dirac theory. This is part of an attempt to understand the strange fact that the velocity operator in the Dirac theory is a spin operator. That is, if we define the velocity operator through the Heisenberg equations of motion,

$$\mathring{\vec{x}} = -\frac{i}{\hbar}[\vec{x}, H],$$

and use the Dirac Hamiltonian,

$$H = c\vec{\alpha}\cdot(\vec{p} - q/c\,\vec{A}) + q\Phi + mc^2\beta,$$

we find

$$\dot{\vec{x}} \equiv \vec{v} = c\vec{\alpha}.$$

In the nonrelativistic Schrödinger theory, with $H = p^2/2m + V$, we find $\vec{v} = \dot{\vec{x}} = \vec{p}/m$. The velocity is a purely spatial operator. In the presence of a magnetic field this is modified,

$$\vec{v} = \frac{1}{m}\vec{\pi} = \frac{1}{m}(\vec{p} - q/c\,\vec{A}),$$

but it is still a purely spatial operator.

To return to the Dirac theory, the velocity operator also appears in the definition of the current,

$$\vec{J} = c\psi^{+}\vec{\alpha}\psi = \psi^{+}\vec{v}_{op}\psi$$

where "op" means "operator". This is logical, if $\rho = \psi^{+}\psi$ is the probability density, then the current represents the flow of probability, so $\vec{J} = \psi^{+}\vec{v}_{op}\psi$ is reasonable. In any case, these definitions for ρ and $\vec{J}$ imply $\frac{\partial\rho}{\partial t} + \nabla\cdot\vec{J} = 0$, assuming ψ satisfies the Dirac equation (including possibly an interaction with the EM field). But what does it mean that $\vec{v}$ is a ~~pseudo~~ purely spin operator?

Let's go back to the Schrödinger theory and review. ~~Start~~ with a spinless particle, and assume the Sch. eqn is

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\nabla - q/c\,\vec{A}\right)^2 \psi + q\Phi\psi .$$

Then we define

$$\rho = \psi^* \psi$$

$$\vec{J} = \mathrm{Re}\,(\psi^* \vec{v}_{op}\,\psi)$$

where $$\vec{v}_{op} = \frac{1}{m}\left(\vec{p} - q/c\,\vec{A}\right) = \frac{1}{m}\left(-i\hbar\nabla - q/c\,\vec{A}\right).$$

$\vec{J}$ looks like an expectation value of $\vec{v}_{op}$ but it is not because we do not integrate over space. Thus, $\vec{J} = \vec{J}(\vec{x},t)$.

The justification of these definitions is that they imply

$$\frac{\partial \rho}{\partial t} + \nabla\cdot\vec{J} = 0 .$$

But this condition alone does not uniquely determine $\vec{J}$, because if we add the curl of any vector field $\vec{F}$,

$$\vec{J} \longrightarrow \vec{J} + \nabla\times\vec{F},$$

then $\nabla\cdot\vec{J} \rightarrow \nabla\cdot\vec{J}$ and the continuity equation still holds. We can measure ρ (the probability density for position measurements), but we cannot directly measure how probability flows.

plausible definition of a

But in the Schrödinger theory for a spinless particle there is no vector $\vec{F}$ that we could use in $\vec{J} \to \vec{J} + \nabla \times \vec{F}$, so we assume that the usual definition of $\vec{J}$ is the correct one.

Now turn to the Pauli theory of a spin-1/2 particle (to take a special case). The wave function is a 2-component spinor,

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix},$$

and the Pauli equation in the presence of an EM field is

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{1}{2m}\left(-i\hbar\nabla - q/c\,\vec{A}\right)^2 \psi \;-\; \vec{\mu}\cdot\vec{B}\,\psi + q\Phi\psi$$

where

$$\vec{\mu} = g\,\frac{q}{2mc}\,\vec{S} = g\,\frac{q}{2mc}\,\frac{\hbar}{2}\,\vec{\sigma}.$$

(Don't confuse the g-factor g with the charge q.)

In this case we define

$$\rho = \psi^\dagger \psi = |\psi_+|^2 + |\psi_-|^2,$$

which is reasonable, since ρ is the probability density of finding the particle, regardless of its spin state (up or down).

As for the current, we can define

$$\vec{J} = \mathrm{Re}\left(\psi^\dagger\, \vec{v}_{op}\, \psi\right),$$

where $\vec{v}_{op} = \frac{1}{m}(-i\hbar\nabla - q/c\,\vec{A})$, which looks just like the definition in the Schrödinger theory of a spinless particle, except for the replacement of ψ^* and ψ by $\psi^\dagger$ and ψ. This $\vec{J}$ is an expectation value insofar as the spin is concerned, but not a full one since we do not integrate over $\vec{x}$. In addition to its plausibility, this definition also causes the continuity eqn to be satisfied, so it would seem to be the correct definition.

Or is it? If we multiply the probability density and current by the charge q we get a charge density and current,

$$\rho' = q\,\psi^{+}\psi = q\rho$$

$$\vec{J}' = q\,\mathrm{Re}(\psi^{+}\vec{v}_{op}\psi) = q\vec{J}$$

where the prime distinguishes charge $\rho', \vec{J}'$ from the probability $\rho, \vec{J}$. Although we cannot measure the flow of probability, electric currents give rise to magnetic fields, which are measurable. So is this the correct $\vec{J}'$, or can we add some term of the form $g\,\nabla\times\vec{F}$?

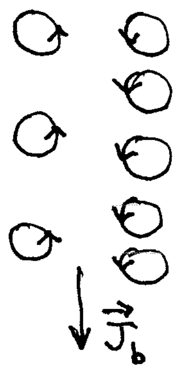
In fact it is plausible that there should be such a term, when the particle has spin. In classical electromagnetic theory, one describes a magnetic material by its magnetization $\vec{M}$, defined as the magnetic dipole moment per unit volume. But a particle with spin has a magnetic moment, and it is reasonable to associate a wave function ψ (which is a spinor) with the magnetization

$$\vec{M} = \psi^{+}\vec{\mu}\psi = g\frac{q}{2mc}\psi^{+}\vec{S}\psi.$$

Also, a nonuniform magnetization (to return to classical EM theory) gives rise to a bound current density

$$\vec{J}_b = c\,\nabla\times\vec{M}\,.$$

When the density of dipoles is not uniform, then the current from the small current loops is not exactly canceled by the current from the neighboring loops:



So, in the case of a particle with spin, we can argue that the charge current should be

$$\vec{J}\,' = q\,\mathrm{Re}\left(\psi^{+}\vec{v}_{op}\psi\right) + c\,\nabla\times\left(\psi^{+}\vec{\mu}\,\psi\right).$$

~~Now,~~ The extra term has no effect on the continuity equation.

Now, does this mean that the probability current should be

$$\vec{J} = \mathrm{Re}\left(\psi^{+}\vec{v}_{op}\psi\right) + \underbrace{\frac{c}{q}\,\nabla\times\left(\psi^{+}\vec{\mu}\,\psi\right)}_{=\,\frac{g\hbar}{4m}\,\nabla\times\left(\psi^{+}\vec{\sigma}\,\psi\right)} = \frac{1}{q}\,\vec{J}\,'$$

(where the last expression applies when $s = 1/2$)? We can argue that it does, after all, the magnetization or bound current produces real magnetic fields which can be measured. If we do this, then we see that the probability current has a spin component in the nonrelativistic Pauli theory, and it is less surprising that in the Dirac theory the probability current should involve the spin.

Now return to the Dirac theory. Let's work with a free particle for simplicity, so $H = c\vec{\alpha}\cdot\vec{p} + mc^2\beta$. Let ψ be a solution of the free particle Dirac equation, which we write in covariant form:

$$i\hbar\,\gamma^\nu \frac{\partial\psi}{\partial x^\nu} = mc\,\psi .$$

Multiply this on the left by $\bar{\psi}\gamma^\mu$ to obtain

$$i\hbar\,\bar{\psi}\gamma^\mu\gamma^\nu\frac{\partial\psi}{\partial x^\nu} = mc\,\bar{\psi}\gamma^\mu\psi = m\,J^\mu, \qquad (*)$$

where we use $J^\mu = c\,\bar{\psi}\gamma^\mu\psi$, the covariant expression for the current. The point is to manipulate the Dirac equation to get an expression for the current.

Similarly, take the Hermitian conjugate of the Dirac equation,

$$-i\hbar\,\frac{\partial\psi^+}{\partial x^\nu}\overset{\gamma^0\gamma^0}{\vee}(\gamma^\nu)^+ = mc\,\psi^+,$$

multiply from the right by γ^0 and insert γ^0's as shown, to get

$$-i\hbar \frac{\partial \bar{\psi}}{\partial x^\nu} \gamma^\nu = mc\bar{\psi}$$

where we use $\psi^\dagger \gamma^0 = \bar{\psi}$ and $\gamma^0 (\gamma^\nu)^\dagger \gamma^0 = \gamma^\nu$. Now multiply this from the right by $\gamma^\mu \psi$ to get

$$-i\hbar \frac{\partial \bar{\psi}}{\partial x^\nu} \gamma^\nu \gamma^\mu \psi = mc\bar{\psi}\gamma^\mu \psi = m J^\mu, \qquad (**)$$

another expression for the current.

Now in (*) use

$$\gamma^\mu \gamma^\nu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} + \frac{1}{2}[\gamma^\mu, \gamma^\nu]$$

$$= g^{\mu\nu} - i\sigma^{\mu\nu}$$

where

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$$

is the generator of Lorentz transformations. Likewise in (**) use

$$\gamma^\nu \gamma^\mu = g^{\nu\mu} - i\sigma^{\nu\mu} = g^{\mu\nu} + i\sigma^{\mu\nu}.$$ Then add the two equations to get

$$i\hbar\, \bar{\psi} g^{\mu\nu} \frac{\partial \psi}{\partial x^\nu} - i\hbar \frac{\partial \bar{\psi}}{\partial x^\nu} g^{\mu\nu} \psi$$

$$+ \hbar\, \bar{\psi} \sigma^{\mu\nu} \frac{\partial \psi}{\partial x^\nu} + \hbar \frac{\partial \bar{\psi}}{\partial x^\nu} \sigma^{\mu\nu} \psi = 2m J^\mu.$$

This can be rearranged into $\rightarrow$ here $p^\mu = i\hbar\partial^\mu$

$$J^\mu = \mathrm{Re}\left[\bar{\psi}\left(\frac{p^\mu}{m}\right)\psi\right] + \frac{\hbar}{2m}\frac{\partial}{\partial x^\nu}\left(\bar{\psi}\sigma^{\mu\nu}\psi\right).$$

This is called the <u>Gordon decomposition</u> of the current. The first term is the <u>convective</u> term, corresponding to the motion of the charged particle, and the second is the <u>magnetization</u> term.

The convective term is the obvious relativistic generalization of $\vec{J} = \mathrm{Re}[\psi^*(\vec{p}/m)\psi]$ that applies in the Schrödinger theory (when there are no magnetic fields). It is also the obvious generalization of the Klein-Gordon current $J^\mu = \mathrm{Re}[\psi^*(p^\mu/m)\psi]$ to the Dirac theory, which we recall suffered from the fact that $J^0 = c\rho$ is not positive definite. The spin term is a relativistic version of the magnetization current in the Pauli theory.

We see that the velocity operator $c\vec{\alpha}$ in the Dirac theory includes both the convective and magnetization contributions which appear in the Pauli theory.