We turn now to free particle solutions of the Dirac equation, following Bjorken and Drell Ch. 3 quite closely. First we review free particle solution in the Pauli theory (the nonrelativistic theory for spin ½ particles). In the Pauli theory, the wave function is a 2-component spinor,

$$\psi = \begin{pmatrix} \psi_+ \\ \psi_- \end{pmatrix},$$

and the free-particle Hamiltonian is $p^2/2m$. This Hamiltonian commutes with both $\vec{S}$ and $\vec{p}$, so we can find simultaneous eigenstates of S_z and $\vec{p}$. For example, a spin-up state of momentum $\vec{p}$ is

$$\psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{i(\vec{p}\cdot\vec{x} - Et)/\hbar},$$

including the time-dependence, where $E = p^2/2m$. If we boost the particle (change $\vec{p}$), we do not have to change the spinor. (In this discussion we have been sloppy in not distinguishing the operator $\vec{p}$ from the eigenvalues $\vec{p}$ — in a moment we will have to be more careful.)

In the Dirac theory, the angular momentum is

$$\vec{J} = \vec{L} + \frac{\hbar}{2}\vec{\Sigma},$$

where $\vec{\Sigma}$ is the Dirac matrix

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \qquad \text{(Dirac-Pauli).}$$

(We will use only the Dirac-Pauli representation in this set of notes.) The free particle Hamiltonian is

$$H = c\,\vec{\alpha}\cdot\vec{p} + mc^2\beta$$

It is easy to see that H commutes with $\vec{p}$ but it does <u>not</u> commute with either $\vec{L} = \vec{x} \times \vec{p}$ or $\vec{S} = \frac{\hbar}{2}\vec{\Sigma}$ separately. Thus we cannot find eigenstates of $\vec{p}$ and S_z in general, unlike the Pauli theory. This is reflected by the fact that when we boost a free particle solution of the Dirac equation, we must change the spinor as well as the spatial part of the wave function.

Previously we found the free particle solution at rest. With $x = (ct, \vec{x})$, these are

$$\psi_0^r(x) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} e^{-imc^2t/\hbar}, \quad \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} e^{-imc^2t/\hbar}, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} e^{+imc^2t/\hbar}, \quad \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} e^{imc^2t/\hbar}$$

"at rest"

$$r = \quad 1 \qquad\qquad 2 \qquad\qquad 3 \qquad\qquad 4$$

Here r is a label of the four solutions. Applying $i\hbar\frac{\partial}{\partial t}$ to ψ we get the energies of these solutions; they are given by

$$i\hbar \frac{\partial}{\partial t} \psi_0^r(x) = \varepsilon_r \, mc^2 \, \psi_0^r(x),$$

where

$$\varepsilon_r = \begin{cases} +1, & r = 1,2 \\ -1, & r = 3,4. \end{cases}$$

The solutions $r=1,2$ have positive energy $+mc^2$, while those with $r=3,4$ have negative energy $-mc^2$. We have as yet no physical interpretation of these solutions, but it turns out that we cannot ignore them or simply declare them to be nonphysical. More on that later, but for now we retain the negative energy solutions and explore their properties.

The spinors that occur in these solutions will be denoted $w^r(0)$, so that

$$w^1(0) = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad w^2(0) = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad w^3(0) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad w^4(0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

These spinors are just "unit vectors" in spin space.

Let us also write the 4-momentum of the particle at rest by

$$\underbrace{p_0^\mu}_{\text{"at rest"}} = \begin{pmatrix} mc \\ \vec{0} \end{pmatrix}$$

so that

$$p_{0\mu} x^\mu = mc^2 t.$$

Then the four solutions at rest can be written

$$\psi_0^r(x) = w^r(0)\, e^{-i\epsilon_r p_{0\mu} x^\mu/\hbar}.$$

To create solutions in some state of motion we apply a boost. Let $\vec{p}$ be a 3-momentum. We define an energy E and a 4-momentum p^μ as a fn. of $\vec{p}$ by

$$E = \sqrt{m^2c^4 + c^2(\vec{p})^2} = E(\vec{p}),$$

$$p^\mu = p^\mu(\vec{p}) = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix}.$$

Notice in particular that by this definition $E > 0$. Then we associate $\vec{p}$ with a direction $\hat{b}$ of a boost and a rapidity λ by

$$\vec{p} = \hat{b}\, p,$$

$$\cosh\lambda = \frac{E}{mc^2} = \gamma = \frac{1}{\sqrt{1-v^2/c^2}}$$

$$\sinh\lambda = \frac{p}{mc} = \gamma\beta = \frac{v/c}{\sqrt{1-v^2/c^2}}$$

$$\tanh\lambda = \beta = v/c.$$

Here β, γ are the usual parameters of special relativity, not the Dirac matrices. Then we introduce the Lorentz transformation (a pure boost) by

$$\Lambda(\vec{p}) = \Lambda(\hat{b}, \lambda),$$

which has the property that

$$p^\mu = \Lambda(\vec{p})^\mu{}_\nu \, p_0^\nu .$$

That is, $\Lambda(\vec{p})$ boosts a classical particle at rest into one with momentum $\vec{p}$ and energy $E > 0$.

The associated spinor matrix is $D(\hat{b}, \lambda)$ which we will also write as $D(\vec{p})$. It is

$$D(\vec{p}) = \begin{bmatrix} \cosh \lambda/2 & (\hat{b}\cdot\vec{\sigma}) \sinh \lambda/2 \\ (\hat{b}\cdot\vec{\sigma}) \sinh \frac{\lambda}{2} & \cosh \frac{\lambda}{2} \end{bmatrix}$$

$$= \cosh \frac{\lambda}{2} \begin{bmatrix} 1 & (\hat{b}\cdot\vec{\sigma}) \overset{\tanh}{\cancel{\sinh}} \frac{\lambda}{2} \\ (\hat{b}\cdot\vec{\sigma}) \overset{\tanh}{\cancel{\sinh}} \frac{\lambda}{2} & 1 \end{bmatrix}.$$

The parameters of this matrix can be expressed as functions of $\vec{p}$ or E. From the hyperbolic trigonometric identities, we have

$$\cosh \frac{\lambda}{2} = \sqrt{\frac{\cosh \lambda + 1}{2}} = \sqrt{\frac{E + mc^2}{2mc^2}}$$

$$\tanh \frac{\lambda}{2} = \frac{\sinh \lambda}{1 + \cosh \lambda} = \frac{cp}{E + mc^2}.$$

Using $p\hat{b} = \vec{p}$, we can write

$$D(\vec{p}) = \sqrt{\frac{E + mc^2}{2mc^2}} \begin{bmatrix} 1 & \frac{c(\vec{p}\cdot\vec{\sigma})}{E + mc^2} \\ \frac{c(\vec{p}\cdot\vec{\sigma})}{E + mc^2} & 1 \end{bmatrix}.$$

Now we boost the solutions at rest. As for the space-time dependence, it becomes

$$e^{-i\varepsilon_r p_{0\mu} x^\mu/\hbar} \longrightarrow e^{-i\varepsilon_r p_{0\mu} (\Lambda^{-1}x)^\mu/\hbar}$$

$$= e^{-i\varepsilon_r (\Lambda p_0)_\mu x^\mu/\hbar} = e^{-i\varepsilon_r p_\mu x^\mu/\hbar},$$

that is, the (rest) 4-momentum p_0^μ is boosted into $p^\mu = (E/c, \vec{p})$, as we expect. As for the spin part, we define

$$w^r(\vec{p}) = D(\vec{p})\, w^r(0).$$

Then the boosted solutions are

$$\psi^r(x) = w^r(\vec{p})\, e^{-i\varepsilon_r p_\mu x^\mu/\hbar}.$$

Since the $w^r(0)$ are just "unit vectors", it means that the $w^r(\vec{p})$ are the columns of $D(\vec{p})$, that is,

$$D(\vec{p}) = \sqrt{\frac{E+mc^2}{2mc^2}} \begin{bmatrix} 1 & \dfrac{c\vec{p}\cdot\vec{\sigma}}{E+mc^2} \\ \dfrac{c\vec{p}\cdot\vec{\sigma}}{E+mc^2} & 1 \end{bmatrix} = \begin{bmatrix} \begin{pmatrix} \\ w^1(\vec{p}) \\ \\ \end{pmatrix} & \cdots & \begin{pmatrix} \\ w^4(\vec{p}) \\ \\ \end{pmatrix} \end{bmatrix},$$

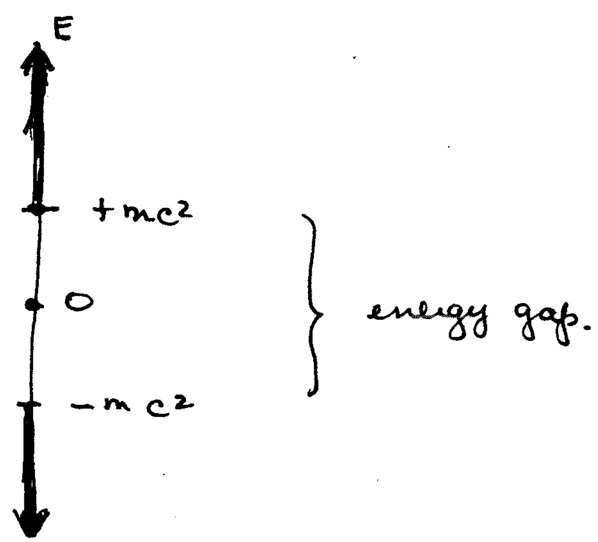
which makes it easy to read off the $w^r(\vec{p})$.

~~the spinors to~~

Let us examine the energy and momentum of these boosted, free particle solutions. Here we use $H = i\hbar \frac{\partial}{\partial t}$ and $\vec{p}_{op} = -i\hbar\nabla$, distinguishing the operator $\vec{p}_{op}$ from the c-number vector $\vec{p}$. As for the energy, we have

$$i\hbar \frac{\partial}{\partial t} \psi^r(x) = \varepsilon_r E \psi^r(x),$$

where recall $E = +\sqrt{m^2c^4 + c^2\vec{p}^{\,2}} > 0$. ~~So r spinors~~ So solutions $r=1,2$ have energy E, while solutions $r=3,4$ have energy $-E$. We see that when we boost a negative energy solution at rest, its energy becomes even more negative. Also, we see that the spectrum of the Dirac free particle Hamiltonian consists of a part $E \geq mc^2$, and a part $E \leq -mc^2$:



There is a gap in the spectrum between $-mc^2$ and $+mc^2$.

As for the momentum of the boosted free particle solutions, we have

$$\vec{p}_{op} \psi^r(x) = -i\hbar\nabla\psi^r(x) = \varepsilon_r \vec{p}\, \psi^r(\vec{x}).$$

For $r=1,2$, the momentum $\vec{p}$ is the same as the parameter of the boost, while for $r=3,4$, it is $-\vec{p}$. When we boost a negative energy solution at rest in the direction $\vec{p}$, it acquires a momentum in the opposite direction (opposite the momentum label of the boost). In a sense, negative energy solutions also have negative momentum.

The spinors $w^r(\vec{p})$ have several important orthonormality and completeness and other relations, which are the spin parts of similar relations satisfied by the full wave function (including the spatial part). The first is

$$(\not{p} - \varepsilon_r mc)\, w_r(\vec{p}) = 0 .$$

Recall that $\not{p} = p_\mu \gamma^\mu$ (the Feynman slash), and p_μ here means the c-number 4-vector, a function of the 3-momentum $\vec{p}$, and not the operator. This is a purely spinor equation with no spatial dependence. It is proved by appeal to the Dirac equation which includes the space-time dependence,

the operator $P^\mu_{op} = i\hbar \partial^\mu$.

$$(\not{P}_{op} - mc)\, \psi^r(x) = \left(i\hbar \gamma^\mu \frac{\partial}{\partial x^\mu} - mc\right) w_r(\vec{p})\, e^{-i\varepsilon_r p_\mu x^\mu/\hbar}$$

$$= \left(\varepsilon_r p_\mu \gamma^\mu - mc\right) w_r(\vec{p})\, e^{-i\varepsilon_r p_\mu x^\mu/\hbar} = 0,$$

or

$$(\varepsilon_r \not{p} - mc)\, w_r(\vec{p}) = 0$$

after cancelling the phase factor. Multiplying this by ε_r and using $\varepsilon_r^2 = 1$ gives the result.

This identity comes in the adjoint version:

$$\bar{w}^r(\vec{p})\,(\not{p} - \varepsilon_r mc) = 0. \qquad \text{To prove it...}$$

Take the Hermitian conjugate of the eqn. above and insert factors of γ^0:

$$(p_\mu \gamma^\mu - \varepsilon_r mc)\, w_r(\vec{p}) = 0$$

$$\Downarrow$$

$$w_r(\vec{p})^+ \underset{\gamma^0\gamma^0}{\wedge} \left(p_\mu (\gamma^\mu)^+ - \varepsilon_r mc\right) \underset{\gamma^0}{\wedge} = 0$$

$$\Downarrow$$

$$\bar{w}_r(\vec{p})\,(p_\mu \gamma^\mu - \varepsilon_r mc) = 0$$

where we use $w_r(\vec{p})^+ \gamma^0 = \bar{w}_r(\vec{p})$, $\gamma^0 (\gamma^\mu)^+ \gamma^0 = \gamma^\mu$ and $(\gamma^0)^2 = 1$.

Next property is the adjoint orthonormality relation,

$$\bar{w}_r(\vec{p})\, w_{r'}(\vec{p}) = \varepsilon_r \delta_{rr'}$$

To prove this, note that since $w_r(0)$ are just unit vectors, then $w_r(0)^+ M w_{r'}(0) = M_{rr'}$ where M is any 4x4 spinor matrix. In particular,

$$w_r(0)^+ \gamma^0 w_{r'}(0) = (\gamma^0)_{rr'} = \varepsilon_r \delta_{rr'}$$

$$= \bar{w}_r(0)\, w_{r'}(0).$$

But

$$\bar{w}_r(\vec{p})\, w_{r'}(\vec{p}) = \bar{w}_r(0)\, D(\vec{p})^{-1} D(\vec{p})\, w_{r'}(0) = \bar{w}_r(0)\, w_{r'}(0).$$

This proves the result.

Next property is the adjoint completeness relation,

$$\sum_r \varepsilon_r \, w_r(\vec{p})\, \bar{w}_r(\vec{p}) = 1 \,,$$

where the product of the column spinor $w_r(\vec{p})$ and the row spinor $\bar{w}_r(\vec{p})$ is interpreted as an outer product or a matrix. To prove this, let us temporarily use Dirac notation for a complete, ~~orthonormal~~ orthonormal basis of spinors, $\{|r\rangle, r=1,2,3,4\}$. Then

$$\gamma^0 = \sum_{rr'} |r\rangle\langle r|\gamma^0|r'\rangle\langle r'| \,.$$

Now let $|r\rangle$ be identified with the unit vectors $w_r(0)$. In this basis γ^0 has components $\varepsilon_r \delta_{rr'} = (\gamma^0)_{rr'}$, so

$$\gamma^0 = \sum_{rr'} w_r(0)\ \varepsilon_r \delta_{rr'}\ w_{r'}(0)^\dagger = \sum_r \varepsilon_r\, w_r(0)\, w_r(0)^\dagger \,.$$

Now multiply from the right by γ^0 to get

$$1 = \sum_r \varepsilon_r\, w_r(0)\, \bar{w}_r(0) \,.$$

means $D(\vec{p})$

Finally, multiply from the left by $D(\Lambda)$ and the right by $D(\Lambda)^{-1}$, to get

$$1 = \sum_r \varepsilon_r\, w_r(\vec{p})\, \bar{w}_r(\vec{p}) \,,$$

which was to be proved.

Finally, there is the Hermitian orthonormality relation. It is

$$w_r(\varepsilon_r \vec{p})^{+} \; w_{r'}(\varepsilon_{r'} \vec{p}) = \frac{E}{mc^2} \, \delta_{rr'} .$$

To remember this it helps to recall the theorem that eigenvectors of a Hermitian operator with distinct eigenvalues are orthogonal. The relation above refers only to spinors, but the complete wave function, including space-time dependence, is $w_r(\vec{p})\, e^{-i\varepsilon_r p_\mu x^\mu/\hbar}$, and its momentum eigenvalue is $\varepsilon_r \vec{p}$. So the ε_r, $\varepsilon_{r'}$ inside the two spinors on the left guarantee that the momentum eigenvalues are equal. For example, if both $r, r' = 1, 2$, then the arguments are $\vec{p}$ and $\vec{p}$, which are equal. If both $r, r' = 3, 4$, then the arguments are $-\vec{p}$ and $-\vec{p}$, which again are equal. But if $r = 1, 2$ and $r' = 3, 4$, or vice versa, then the momentum values are opposite, and the spinors are orthogonal. As for the factor E/mc^2 on the right, it appears because the boosts $D(\vec{p})$ are not unitary, so the norm of the spinors changes when we boost them. In fact, this is the usual relativistic factor $\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$, which, as explained previously, compensates for the Lorentz contraction of the volume element $d^3\vec{x}$ to make the normalization integral

$$\int d^3\vec{x} \quad \psi^{+}\psi$$

invariant under a Lorentz transformation. That is, the Lorentz transformation, including both spatial and spin parts, is unitary. We will use this Hermitian orthonormality relation more than the others.

Actually, the spinors $w^r(\vec{p})$ are not general enough for our purposes, because the spin is polarized in the $\pm z$-direction in the rest frame. That is, the spinors $w^r(0)$ are eigenspinors of

$$\Sigma_3 = \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

with eigenvalues $+1, -1, +1, -1$ for $r = 1,2,3,4$. We will have need for spinors that are polarized in an arbitrary direction in their rest frame.

Before we do that, however, we note that after we boost the spinors to get $w^r(\vec{p})$, they are no longer eigenstates of Σ_3 (or any other component of $\vec{\Sigma}$). This is a reflection of the fact that the spin $\vec{\Sigma}$ does not commute with the Dirac free particle Hamiltonian $H = c\vec{\alpha}\cdot\vec{p} + mc^2\beta$, so a simultaneous eigenbasis of H and any component of $\vec{\Sigma}$ does not exist. This is in contrast to the Pauli theory, where $\binom{1}{0} e^{i\vec{p}\cdot\vec{x}/\hbar}$ is an eigenstate of S_z and $H = p^2/2m$ for any value of $\vec{p}$.

Nevertheless, in the Dirac theory it is possible to organize free particle solutions by the eigenvalue of Σ_3 or $\hat{n}\cdot\vec{\Sigma}$ (in any direction) in the <u>rest frame</u>.

Let $\hat{s}$ be a unit vector in the rest frame. We promote this into a 4-vector in the rest frame by writing,

$$s_0^\mu = (0, \hat{s})$$

"rest frame"

The 4-momentum in the rest frame is

$$p_0^\mu = (mc, \vec{0}),$$

so

$$p_0^\mu p_{0\mu} = m^2c^2$$

$$s_0^\mu s_{0\mu} = -1$$

$$p_0^\mu s_{0\mu} = 0$$

The momentum and spin 4-vectors are orthogonal.

Now let $u(p_0, s_0)$ be a spinor of positive energy $+mc^2$ at rest, polarized in the direction $\hat{s}$. It satisfies

$$\left.\begin{aligned}(\hat{s}\cdot\vec{\Sigma})\, u(p_0,s_0) &= u(p_0,s_0) \\ (\not{p}_0 - mc)\, u(p_0,s_0) &= 0,\end{aligned}\right\}$$

where the second condition means that the energy is $+mc^2$ $(\varepsilon_r = +1)$. By taking two u-type spinors polarized in the directions $\pm\hat{s}$, we get a complete set of positive energy spinors. These have 4-spinors $\pm s_0^\mu$. If $\hat{s} = \pm\hat{z}$, we get the spinors $w^r(0)$, $r=1,2$. The equations above specify the spinors $u(p_0,s_0)$ to within a phase, which will not be important in our applications.

Having defined $u(p_0, s_0)$ in the rest frame, we now boost these spinors by $\Lambda(\vec{p})$ or $D(\vec{p})$, to define

$$u(p,s) = D(\vec{p})\, u(p_0, s_0),$$

where

$$p^\mu = \Lambda(\vec{p})^\mu{}_\nu \, p_0^\nu = (E/c, \vec{p})$$

and $\quad s^\mu = \Lambda(\vec{p})^\mu{}_\nu \, s_0^\nu .$

Because Minkowski scalar products are invariant under Lorentz transformations, we have

$$\left.\begin{aligned} p^\mu p_\mu &= m^2c^2 \\ s^\mu s_\mu &= -1 \\ p^\mu s_\mu &= 0 \end{aligned}\right\}.$$

Then

$$(\not{p} - mc)\, u(p,s) = 0.$$

Similarly, we define a spinor $v(p_0, s_0)$ in the rest frame to have <u>negative</u> energy $-mc^2$, and spin polarized in the <u>opposite</u> direction to $\hat{s}$, where $s_0^\mu = (0, \hat{s})$ as with the u-type spinors. That is,

$$(\not{p}_0 + mc)\, v(p_0, s_0) = 0$$

$$(\hat{s} \cdot \vec{\Sigma})\, v(p_0, s_0) = -\, v(p_0, s_0)$$

↑ note sign.

We can define anything we want, but why do we define the v-type spinor to have spin opposite the direction of $\hat{s}$? It is because these spinors have negative energy, and a opposite that of the momentum label (that is, with "negative momentum.") So, by convention, we will make their spin also opposite the spin label (that is, they have "negative spin.")

Then we boost these spinors, writing

$$v(p,s) = D(\vec{p})\, v(p_0, s_0),$$

so that

$$(\not{p} + mc)\, v(p,s) = 0.$$

The spinors $u(p,s)$, $u(p,s')$, $v(p,s)$, $v(p,s')$, where s, s' correspond to $\pm \hat{s}$ (some choice of unit vector) in the rest frame, span the space of all spinors. The u- and v-spinors satisfy orthonormality relations (adjoint and Hermitian) that are the analogs of those discussed above for the w-type spinors. These are:

$$\left.\begin{matrix} (\not{p} - mc)\, u(p,s) = 0 \\ (\not{p} + mc)\, v(p,s) = 0 \end{matrix}\right\}, \qquad \left.\begin{matrix} \bar{u}(p,s)\,(\not{p} + mc) = 0 \\ \bar{v}(p,s)\,(\not{p} + mc) = 0 \end{matrix}\right\};$$

$$\bar{u}(p,s)\, u(p,s') = \delta_{ss'}$$

$$\bar{u}(p,s)\, v(p,s') = 0$$

$$\bar{v}(p,s)\, u(p,s') = 0$$

$$\bar{v}(p,s)\, v(p,s') = -\delta_{ss'}$$

where it is understood that s, s' are 4-vectors that are boosted versions of $\pm \hat{s}$ (a choice of polarization vector) in the rest frame. Next, the adjoint completeness relation:

$$\sum_{s} u(p,s)\,\bar{u}(p,s) - v(p,s)\,\bar{v}(p,s) = 1$$

where s takes on 2 values corresponding to a choice of $\pm\hat{s}$ in the rest frame.

Finally, the Hermitian orthonormality relations,

$$u(p,s)^{+}\, u(p,s') = \frac{E}{mc^2}\,\delta_{ss'}$$

$$u(p,s)^{+}\, v(\tilde{p},s') = 0$$

$$v(\tilde{p},s)^{+}\, u(p,s') = 0$$

$$v(p,s)^{+}\, v(p,s') = \frac{E}{mc^2}\,\delta_{ss'}$$

where s, s' take on two values as above, and where $\tilde{p}^{\mu} = (E/c, -\vec{p})$ when $p^{\mu} = (E/c, \vec{p})$.

Finally, we mention energy projection operators. If we have an arbitrary spinor, it is often convenient to project out the positive and negative energy components. The projectors $\Pi_{\pm}$ that do this are defined by

$$\Pi_{\pm} = \frac{\not{p} \overset{\mp}{\pm} mc}{2mc},$$

so that

$$\Pi_{+}\, u(p,s) = u(p,s) \qquad \Pi_{-}\, u(p,s) = 0$$

$$\Pi_{+}\, v(p,s) = 0 \qquad \Pi_{+}\, v(p,s) = v(p,s)$$

Also, we have

$$\Pi_+^2 = \Pi_+$$

$$\Pi_-^2 = \Pi_-$$

$$\Pi_+ + \Pi_- = 1 .$$

Bjorken and Drell also discuss spin projectors, but we will defer discussing these until when (and if) we need them.