Let's compute $D(\Lambda)$ explicitly for pure rotations and pure boosts.

Rotations first. In this case $\lambda_i = \theta_{0i} = 0$, so

$$\frac{1}{4}\theta_{\alpha\beta}\sigma^{\alpha\beta} = \frac{1}{4}\theta_{ij}\sigma^{ij}.$$

As for σ^{ij}, it is

$$\sigma^{ij} = \frac{i}{2}[\gamma^i,\gamma^j]$$

and

$$[\gamma^i,\gamma^j] = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} - (i\leftrightarrow j)$$

$$= \begin{pmatrix} -\sigma_i\sigma_j & 0 \\ 0 & -\sigma_i\sigma_j \end{pmatrix} - (i\leftrightarrow j)$$

$$= -2i\,\varepsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix},$$

where we work either in the D-P or Weyl representation (the matrices $\vec{\gamma}$ are the same in both reps. ~~See~~ Now we define a new set of Dirac matrices,

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}_{\text{D-P or Weyl}},$$

so $[\gamma^i,\gamma^j] = -2i\,\varepsilon_{ijk}\,\sigma_k$ and $\sigma^{ij} = \varepsilon_{ijk}\Sigma_k$.

Beware, Bjorken and Drell write $\vec{\sigma}$ for $\vec{\Sigma}$, thereby confusing the 2x2 Pauli matrices $\vec{\sigma}$ with the 4x4 Dirac matrices $\vec{\Sigma}$.

Now the D-matrix for a pure rotation is

$$D(\hat{n},\theta) = e^{-\frac{i}{4}\theta_{ij}\sigma^{ij}} = e^{-\frac{i}{4}\varepsilon_{ijk}\theta_k\,\varepsilon_{ij\ell}\Sigma_\ell}$$

$$= e^{-\frac{i}{2}\theta_k\Sigma_k} = e^{-\frac{i}{2}\vec{\theta}\cdot\vec{\Sigma}} = e^{-\frac{i}{2}\theta\hat{n}\cdot\vec{\Sigma}},$$

where we use $\theta_{ij} = \varepsilon_{ijk}\theta_k$, $\vec{\theta} = \theta\hat{n}$, and $\varepsilon_{ijk}\varepsilon_{ij\ell} = 2\delta_{k\ell}$. However, since $\vec{\Sigma}$ is block-diagonal, the exponential is easily expressed in terms of the spin-½ rotation matrices,

$$U(\hat{n},\theta) = e^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}} = \cos\frac{\theta}{2} - i(\hat{n}\cdot\vec{\sigma})\sin\frac{\theta}{2},$$

$$D(\hat{n},\theta) = e^{-i\frac{\theta}{2}\hat{n}\cdot\vec{\Sigma}} = \left(\begin{array}{c|c} U(\hat{n},\theta) & 0 \\ \hline 0 & U(\hat{n},\theta) \end{array}\right)$$

D-P or Weyl.

Pure Rotations

Now pure boosts for which $\theta_{ij} = 0$, $\vec{\theta} = 0$, $\theta_{0i} = \lambda_i \neq 0$.

Then

$$\frac{1}{4}\theta_{\alpha\beta}\sigma^{\alpha\beta} = \frac{1}{4}\left(\underbrace{\theta_{0i}\sigma^{0i} + \theta_{i0}\sigma^{i0}}_{\text{equal}}\right) = \frac{1}{2}\lambda_i\sigma^{0i}.$$

But

$$\sigma^{0i} = \frac{i}{2}[\gamma^0,\gamma^i] = \frac{i}{2}(\gamma^0\gamma^i - \gamma^i\gamma^0) = \frac{i}{2}(\beta\cdot\beta\alpha_i - \beta\alpha_i\beta)$$

$$= \frac{i}{2}(2\alpha_i) = i\alpha_i$$

So the boost matrix is

$$D(\hat{b},\lambda) = e^{-\frac{i}{2}\lambda_i\sigma^{0i}} = e^{\frac{1}{2}\vec{\lambda}\cdot\vec{\alpha}} = e^{\frac{\lambda}{2}\hat{b}\cdot\vec{\alpha}}.$$

We see that the generators of boosts are the components of $\vec{\alpha}$, essentially the velocity operator. $\vec{\alpha}$ is block-diagonal in the Weyl representation, which makes the exponentiation easier:

$$\vec{\alpha} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & -\vec{\sigma} \end{pmatrix}_{\text{Weyl}},$$

so $e^{\frac{\lambda}{2}\hat{b}\cdot\vec{\alpha}}$ can be expressed in terms of the 2×2 matrices defined by

$$V(\hat{b},\lambda) = e^{\frac{\lambda}{2}\hat{b}\cdot\vec{\sigma}} = \cosh\frac{\lambda}{2} + (\hat{b}\cdot\vec{\sigma})\sinh\frac{\lambda}{2},$$

$$V(\hat{b},\lambda)^{-1} = e^{-\frac{\lambda}{2}\hat{b}\cdot\vec{\sigma}} = \cosh\frac{\lambda}{2} - (\hat{b}\cdot\vec{\sigma})\sinh\frac{\lambda}{2}.$$

The V-matrices are like the U-matrices, but with an imaginary angle. Then

$$D(\hat{b},\lambda) = e^{\frac{\lambda}{2}\hat{b}\cdot\vec{\alpha}} = \left(\begin{array}{c|c} V(\hat{b},\lambda) & 0 \\ \hline 0 & V(\hat{b},\lambda)^{-1} \end{array}\right)_{\text{Weyl}}.$$

To get the boost in the D-P representation, we can either exponentiate the 4×4 matrices in the D-P representation, or change the basis according to

$$W^\dagger X_{Weyl} W = X_{DP},$$

where X is any Dirac matrix and

$$W = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

This gives

$$D(\hat{b}, \lambda) = \begin{pmatrix} \cosh\frac{\lambda}{2} & (\hat{b}\cdot\vec{\sigma}) \sinh\frac{\lambda}{2} \\ (\hat{b}\cdot\vec{\sigma}) \sinh\frac{\lambda}{2} & \cosh\frac{\lambda}{2} \end{pmatrix}_{D-P}.$$

Finally, in view of the fact that an arbitrary proper Lorentz transformation can be written $\Lambda = RB$ where R is a proper rotation and B is a boost, we can use $D(\Lambda) = D(R) D(B)$ to construct $D(\Lambda)$ for an arbitrary Lorentz transformation. This completes the solution of the problem of finding $D(\Lambda)$ for proper L.T.'s. The case of parity (an improper L.T.) will be considered a little later.

The matrices $D(\Lambda)$ have the following properties. First, they are not unitary, in general, so

$$D(\Lambda)^{-1} \neq D(\Lambda)^\dagger.$$

They _are_ unitary in the case of pure rotations, but not otherwise. In fact they are Hermitian for pure boosts, and have no particular symmetry in the general case. ⑤

We usually say unitary transformations are needed to implement symmetry operations, in order to ~~preserve~~ preserve probabilities. So why are the $D(\Lambda)$ not unitary?

The answer is that the $D(\Lambda)$ only implement the spin part of the L.T., but there is a spatial part, too. Since the probability density is $\rho = \psi^+\psi$ in the Dirac theory, a normalized wave function satisfies

$$1 = \int d^3\vec{x} \quad \psi^+(x)\,\psi(x) \qquad , \quad x = (ct, \vec{x}).$$

Under a Lorentz transformation, the volume element $d^3\vec{x}$ is not invariant, but rather scales by a factor of $\gamma = 1/\sqrt{1-v^2/c^2}$, due to the Lorentz contraction. Similarly, the spin part of the Lorentz transformation of ψ guarantees that $\psi^+\psi$ is not an invariant, either (it is the time-component of a 4-vector), instead it also acquires a factor of γ upon doing a Lorentz transformation, which cancels the γ coming from the volume element. Thus, the normalization integral is invariant, but $\psi^+\psi$ is not. That is, the overall transformation is unitary, even if the spin part is not. This discussion has been rather imprecise and lacking in details because a proper understanding of probability conservation in a relativistic theory must take into account the relativity of simultaneity. Nevertheless, we will see the factor of γ appear when we consider the transformation of spinors for free particles under L.T.'s later.

Next we develop the ~~conjugation~~ properties of various Dirac matrices under conjugation by $\beta = \gamma^0$. These will be put to immediate use in the study of the covariance of the 4-current J^μ.

These properties are summarized here:

$$\boxed{\begin{array}{lll} 1) & \gamma^0 \gamma^\mu \gamma^0 & = (\gamma^\mu)^+ \\ 2) & \gamma^0 \sigma^{\mu\nu} \gamma^0 & = (\sigma^{\mu\nu})^+ \\ 3) & \gamma^0 D(\Lambda)^{-1} \gamma^0 & = D(\Lambda)^+ \end{array}}$$

Notice item 3), which shows that although $D(\Lambda)^{-1} \neq D(\Lambda)^+$, nevertheless there is a simple relationship between the two matrices.

The proof of 1) is straightforward:

$$\mu = 0: \qquad \gamma^0\gamma^0\gamma^0 = \gamma^0 = (\gamma^0)^+$$

$$\mu = i: \qquad \gamma^0\gamma^i\gamma^0 = -\gamma^0\gamma^0\gamma^i = -\gamma^i = (\gamma^i)^+$$

So is the proof of 2):

$$\gamma^0 \sigma^{\mu\nu} \gamma^0 = \frac{i}{2}\gamma^0 \Big(\gamma^\mu \underset{\gamma^0\gamma^0}{\wedge} \gamma^\nu - \gamma^\nu \underset{\gamma^0\gamma^0}{\wedge} \gamma^\mu\Big)\gamma^0 = \frac{i}{2}\left[(\gamma^\mu)^+, (\gamma^\nu)^+\right] =$$

$$-\frac{i}{2}[\gamma^\mu,\gamma^\nu]^+ = \left(\frac{i}{2}[\gamma^\mu,\gamma^\nu]\right)^+ = (\sigma^{\mu\nu})^+.$$

Finally, for property 3) let Λ be infinitesimal, so that

$$D(\Lambda) = 1 - \frac{i}{4}\theta_{\mu\nu}\sigma^{\mu\nu}$$

$$D(\Lambda)^{-1} = 1 + \frac{i}{4}\theta_{\mu\nu}\sigma^{\mu\nu}$$

$$D(\Lambda)^+ = 1 + \frac{i}{4}\theta_{\mu\nu}(\sigma^{\mu\nu})^+.$$

Then property 3) (for infinitesimal L.T.'s) follows immediately from property 2). But if 3) is true for any two L.T.'s Λ_1, Λ_2, then it is true for the product $\Lambda_1\Lambda_2$. Therefore by building up finite L.T.'s as the limit of the product of a large number of infinitesimal transformations, it follows that 3) is true for any <u>proper</u> L.T. Below we will show that it is true also for parity (an improper L.T.).

Now we are set to show the covariance of the Dirac current $J^\mu = (c\rho, \vec{J})$, where

$$J^0 = c\rho = c\,\psi^+\psi = c\,\psi^+\gamma^0\gamma^0\psi$$

$$\vec{J} = c\,\psi^+\vec{\alpha}\psi = c\,\psi^+\gamma^0\vec{\gamma}\,\psi.$$

The product $\psi^+(x)\gamma^0$ is of frequent occurrence in the Dirac theory, so we give it a name and a special notation:

$\bar{\psi}(x) \equiv \psi^{+}(x)\gamma^{0}$ = the adjoint spinor.

Notice that $\bar{\psi}$, like ψ^{+} is a row spinor (if we think of ψ as a column spinor).

In terms of this notation, we have

$$\boxed{J^{\mu}(x) = c\bar{\psi}(x)\gamma^{\mu}\psi(x)}$$

which is regarded as the covariant expression for the current. To find how this transforms under L.T.'s, we use the transformation law for $\psi(x)$, $\psi(x) \rightarrow D(\Lambda)\psi(\Lambda^{-1}x)$, to derive that for $\bar{\psi}(x)$:

$$\bar{\psi}(x) = \psi^{+}(x)\gamma^{0} \xrightarrow{\Lambda} \psi^{+}(\Lambda^{-1}x)D(\Lambda)^{+}\gamma^{0}$$

$$= \psi^{+}(\Lambda^{-1}x)\gamma^{0}\gamma^{0}D(\Lambda)^{+}\gamma^{0} = \bar{\psi}(\Lambda^{-1}x)D(\Lambda)^{-1},$$

where we use property 3) above. Thus, the current transforms according to

$$J^{\mu}(x) \xrightarrow{\Lambda} c\,\bar{\psi}(\Lambda^{-1}x)D(\Lambda)^{-1}\gamma^{\mu}D(\Lambda)\psi(\Lambda^{-1}x)$$

$$= c\,\bar{\psi}(\Lambda^{-1}x)\,\Lambda^{\mu}{}_{\nu}\gamma^{\nu}\psi(\Lambda^{-1}x) = \Lambda^{\mu}{}_{\nu}J^{\nu}(\Lambda^{-1}x).$$

This is precisely how a 4-vector field should transform under an (active) L.T, for example, it is how the electromagnetic potential A^{μ} transforms. Compare it to the transformation law for electric fields under spatial rotations given above.

We are collecting some transformation laws for different types of fields. Here is a summary, where we include some new ones, too.

Spinor	$\psi(x)$	$\xrightarrow{\Lambda}$	$D(\Lambda)\,\psi(\Lambda^{-1}x)$	$\equiv \psi'(x)$
Adjoint	$\bar{\psi}(x)$	$\xrightarrow{\Lambda}$	$\bar{\psi}(\Lambda^{-1}x)\,D(\Lambda)^{-1}$	$\equiv \bar{\psi}'(x)$
Vector	$J^{\mu}(x)$	$\xrightarrow{\Lambda}$	$\Lambda^{\mu}{}_{\nu}\, J^{\nu}(\Lambda^{-1}x)$	$\equiv J'^{\mu}(x)$
Tensor	$T^{\mu\nu}(x)$	$\xrightarrow{\Lambda}$	$\Lambda^{\mu}{}_{\alpha}\,\Lambda^{\nu}{}_{\beta}\, T^{\alpha\beta}(\Lambda^{-1}x)$	$\equiv T'^{\mu\nu}(x)$
Scalar	$S(x)$	$\xrightarrow{\Lambda}$	$S(\Lambda^{-1}x)$	$\equiv S'(x)$
Pseudo-scalar	$K(x)$	$\xrightarrow{\Lambda}$	$(\det\Lambda)\, K(\Lambda^{-1}x)$	$\equiv K'(x)$
Pseudo-vector	$A^{\mu}(x)$	$\xrightarrow{\Lambda}$	$(\det\Lambda)\,\Lambda^{\mu}{}_{\nu}\, A^{\nu}(\Lambda^{-1}x)$	$\equiv A'^{\mu}(x)$

Here $J^{\mu}(x)$ is just one example of a vector field, and $T^{\mu\nu}(x)$ is any second-rank tensor. Can such tensors be constructed out of the quantum theory of relativistic particles? Yes, for the Dirac particle there is the antisymmetric tensor

$$T^{\mu\nu}(x) = \bar{\psi}(x)\,\sigma^{\mu\nu}\,\psi(x).$$

It is left as an exercise to show that this does transform as a tensor.

As for the scalar $S(x)$, note the similarity to the transformation law for scalar functions under ordinary rotations, for example, the wave function $\psi(\vec{r})$ for a spinless particle, $\psi(\vec{r}) \xrightarrow{R} \psi(R^{-1}\vec{r})$. Are there Lorentz scalars coming from quantum theory? Yes, the Klein-Gordon wave function $\psi_{KG}(x)$ is an example. It is a scalar because it only has one component, but it also transforms as a scalar under L.T.'s,

$$\psi_{KG}(x) \xrightarrow{\Lambda} \psi_{KG}(\Lambda^{-1}x).$$

The transformation law is simpler than that for a Dirac wavefunction because there is no spin matrix $D(\Lambda)$. This transformation law makes the Klein-Gordon current a valid 4-vector,

$$J^{\mu}_{KG}(x) = \frac{i\hbar}{2m}\left(\psi^{*}_{KG}(x)\,\partial^{\mu}\psi_{KG}(x)\right) + \text{c.c},$$

since ∂^{μ} acting on a scalar transforms as a 4-vector. Of course this doesn't cure the Klein-Gordon problem that J^{0}_{KG} is not positive definite.

The Dirac theory also produces a scalar. It is $\bar{\psi}(x)\psi(x)$ (not $\psi^{\dagger}(x)\psi(x)$, which is the 0-component of a 4-vector). The proof that $\bar{\psi}\psi$ is a scalar will be left as an exercise.

A ~~pse~~ pseudoscalar transforms as a scalar under proper Lorentz transformations, but under improper ones it acquires a ~~--~~ sign given by $\det \Lambda$. In the following the only improper L.T. we will consider is parity, or products of parity times proper L.T.'s. For simplicity we will leave time-reversal out of the picture. With this understanding, $\det\Lambda = +1$ for proper L.T.'S and $\det\Lambda = -1$ for improper ones.

The classical L.T. corresponding to parity is the matrix

$$P = \begin{pmatrix} +1 & & & 0 \\ & -1 & & \\ & & -1 & \\ 0 & & & -1 \end{pmatrix},$$

in other words, it is the <u>spatial</u> <u>inversion</u> operation. We need to find the corresponding (parity) operator acting on Dirac wave functions. In the non-relativistic theory, parity is a purely spatial operation, so we might guess that the transformation law is $\psi(x) \xrightarrow{P} \psi(P^{-1}x)$, that is, $\psi(\vec{x},t) \xrightarrow{P} \psi(-\vec{x},t)$, without any effect on spin. It turns out, however, that this does not work. Instead, we must write

$$\psi(\vec{x},t) \xrightarrow{P} D(P)\psi(-\vec{x},t),$$

where $D(P)$ is a spin matrix to be determined.

By requiring that free particle solutions be mapped into other free particle solutions by parity, we obtain the same requirement on $D(P)$ that we obtained earlier for $D(\Lambda)$, $\Lambda =$ proper. That is,

$$D(P)^{-1}\gamma^\mu D(P) = P^\mu{}_\nu \gamma^\nu .$$

This implies:

$$\mu=0 \qquad D(P)^{-1}\gamma^0 D(P) = \gamma^0 = (\gamma^0)^+$$
$$\mu=i \qquad D(P)^{-1}\gamma^i D(P) = -\gamma^i = (\gamma^i)^+$$

Now since $\gamma^0\gamma^\mu\gamma^0 = (\gamma^\mu)^+$, we see that $D(P) = e^{i\alpha}\gamma^0$ is a solution of this equation, any α. (And we see that $D(P)=1$ will <u>not</u> work; in the relativistic theory parity <u>must</u> involve the spin. This is another indication of the more intimate coupling between spatial and spin degrees of freedom in the relativistic theory.)

The classical matrix P satisfies certain relationships, including

$$P\,R(\hat{n},\theta) = \Lambda(\hat{n},\theta)\,P \qquad \text{(pure rotation)}$$
$$P\,B(\hat{b},\lambda) = B(\hat{b},-\lambda)\,P \qquad \text{(pure boosts)}$$
$$P^2 = I$$

If we require that $D(\Lambda)$ form a representation of the extended Lorentz group containing ~~parit~~ the spatial inversion (still omitting time-reversal, however) then we should have

$$D(P)^2 = 1$$
$$D(P)\,D(\hat{n},\theta) = D(\hat{n},\theta)\,D(P)$$

$$\text{and} \quad D(P)\, D(\hat{b},\lambda) = D(\hat{b},-\lambda)\, D(P).$$

The requirement $D(P)^2 = 1$ is satisfied if we take $e^{i\alpha} = \pm 1$, so $D(P) = \pm\gamma^0$. The $\pm$ choice is purely convention.

Making the choice $D(P) = +\gamma^0$ means that the <u>intrinsic</u> <u>parity</u> of the electron is $+1$. It follows (although we can't get into that yet) that the intrinsic parity of the positron is -1. But no physics would change if we made the opposite convention. For any fermion, the intrinsic parities of the particle and antiparticle are opposite, but it is a matter of convention which is which. We'll choose $D(P) = \gamma^0$ for electrons.

Now since $[\gamma^0, \sigma^{ij}] = 0$, it follows that

$$\gamma^0 D(\hat{n},\theta) = D(\hat{n},\theta)\gamma^0,$$

and since $\gamma^0 \alpha_i = -\alpha_i \gamma^0$, we have

$$\gamma^0 D(\hat{b},\lambda) = D(\hat{b},-\lambda)\gamma^0.$$

It's logical that the velocity operator $= c\vec{\alpha}$ should change sign under parity, and it does. As a result, boosts go into their inverse under conjugation by γ^0.

It is easy to show that true vectors such as $J^\mu(x) = c\,\bar{\Psi}(x)\gamma^\mu\Psi(x)$ transform as shown in the table above under both proper and improper Lorentz transformations (i.e. parity), as does the scalar $S(x) = \bar{\Psi}(x)\Psi(x)$. To construct pseudoscalars and vectors, we need a new Dirac matrix, defined as

$$\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

The 5 subscript is not a space-time index (obviously), it's just a way of inventing a notation for a new Dirac matrix without using up another letter of the Greek alphabet. γ_5 has the following properties:

1) $\gamma_5 = (\gamma_5)^\dagger$

2) $\gamma_5^2 = 1$

3) $\{\gamma_5, \gamma^\mu\} = 0$ hence $\{\gamma_5, D(P)\} = 0$

4) $[\gamma_5, \sigma^{\mu\nu}] = 0$ hence $[\gamma_5, D(\Lambda)] = 0$, when Λ = proper.

We'll just prove property 3). Take the case $\mu = 2$.

$$\gamma_5\gamma^2 = i\gamma^0\gamma^1\gamma^2\gamma^3\gamma^2 = -i\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3$$

$$= -i\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 = -\gamma^2\gamma_5$$

and similarly for other μ. Since $D(P) = \gamma^0$, it follows that $\{\gamma_5, D(P)\} = 0$. Similarly, it's easy to prove $[\gamma_5, \sigma^{\mu\nu}] = 0$, hence $[\gamma_5, D(\Lambda)] = 0$ when $D(\Lambda) = 1 - \frac{i}{4}\theta_{\alpha\beta}\sigma^{\alpha\beta}$, an infinitesimal proper L.T. But if $[\gamma_5, D(\Lambda)] = 0$ for Λ_1 and Λ_2, then it's true for $\Lambda_1\Lambda_2$. So, by building up finite proper L.T.'s out of infinitesimal ones, $[\gamma_5, D(\Lambda)] = 0$ is true for all proper

L.T.'s. We can summarize by writing

$$\gamma_5 D(\Lambda) = (\det \Lambda)\; D(\Lambda)\, \gamma_5 \,,$$

when $\Lambda =$ proper or proper $\times$ $D(P)$.

Now it is easy to construct a pseudoscalar and pseudovector in the Dirac theory. These are

$$K(x) = \bar{\Psi}(x)\, \gamma_5\, \psi(x) \qquad \text{pseudo scalar}$$

$$A^\mu(x) = \bar{\Psi}(x)\, \gamma_5\, \gamma^\mu\, \psi(x) \qquad \text{pseudovector.}$$

It's easy to prove that these have the right transformation laws.

The various fields with various transformation properties under Lorentz transformations are needed to construct Lorentz invariant Lagrangians to model experimental data showing that interactions do or do not respect various symmetries. It is believed that all interactions are invariant under proper Lorentz transformations, at least at scales where gravitational effects are unimportant, but it is known that some interactions do not respect parity or time-reversal. For example, at low energies the weak interaction Lagrangian involves the difference between a vector and a pseudovector (the "V-A theory"), which is responsible for parity violation.

The various fields with the various transformation properties are related to the bilinear covariants, which we now describe. These are associated with the algebra generated by the Dirac matrices γ^μ. The algebra is defined as the set of all matrices that can be formed by multiplying the γ^μ and taking linear combinations. It is the space of all polynomials that can be constructed from the γ^μ.

The case of the Pauli matrices is simpler, so look at it first. Start with the 3 pauli matrices $\vec{\sigma}$. Since $\sigma_i^2 = 1$ (any i), the identity belongs to the algebra, ie the algebra contains $(1, \vec{\sigma})$. As for quadratic expressions involving the σ_i, they can always be reduced to linear expressions, using the identity

$$\sigma_i \sigma_j = \delta_{ij} + i\,\epsilon_{ijk}\,\sigma_k .$$

So the algebra contains all linear (1st degree) polynomials in $\vec{\sigma}$ (ie, all linear combinations of $(1, \vec{\sigma})$). However, $(1, \vec{\sigma})$ span the space of all 2×2 matrices, so the algebra generated by $\vec{\sigma}$ is all 2×2 matrices.

As for the Dirac matrices, since $(\gamma^\mu)^2 = \pm 1$, the algebra contains $(1, \gamma^\mu)$ (at least 1st degree polynomials in γ^μ). The quadratic monomial $\gamma^\mu\gamma^\nu$ looks like 16 matrices, but actually only 6 are independent, because of the anticommutator $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$ (time identity matrix).

The antisymmetric part is captured by $\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu]$, which has 6 independent components. As for cubic monomials in γ, say, $\gamma^\mu\gamma^\nu\gamma^\alpha$, these can be reduced to 1st degree polynomials if any of the indices μ, ν, α are equal. For example, $\gamma^2\gamma^3\gamma^2 = -\gamma^2\gamma^2\gamma^3 = \gamma^3$. So if all indices are distinct, then one of the 4 indices 0,1,2,3 must be left out, so there are actually 4 independent cubic monomials in γ that cannot be reduced. In fact, these are given by $\gamma^\mu\gamma_5$, where μ indicates the one ~~value~~ index that is omitted from the cubic monomial. For example, if $\mu=2$,

$$\gamma^2\gamma_5 = i\,\gamma^2\gamma^0\gamma^1\gamma^2\gamma^3 = +i\,\gamma^0\gamma^1\gamma^2\gamma^2\gamma^3 = -i\,\gamma^0\gamma^1\gamma^3.$$

Finally, a quartic monomial $\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta$ can be reduced to lower degree unless all 4 indices are distinct, in which case the result is proportional to γ_5. So we get the table of basis matrices generated by γ^μ

Name	Matrices	# Components
Scalar	1	1
Vector	γ^μ	4
Tensor	$\sigma^{\mu\nu}$	6
Pseudo vector	$\gamma_5\gamma^\mu$	4
Pseudo scalar	γ_5	1
Sum		16

So 16 matrices are generated ~~asmus~~ by multiplying the γ^μ. These are linearly independent and so span the space of all 4x4 matrices, and they have different transformation properties ~~when~~ under L.T.'s (when sandwiched between $\bar{\psi}$ and ψ).

One final topic concerns the angular momentum of the Dirac electron. We defined the angular momentum last semester as the generator of rotations, and we now know how to rotate Dirac particles, so we can calculate the angular momentum. Under a rotation $\Lambda = R(\hat{n},\theta)$, the Dirac ψ transforms according to

$$\psi(\vec{x}) \xrightarrow{R(\hat{n},\theta)} D(\hat{n},\theta)\,\psi\,\left(R(\hat{n},\theta)^{-1}\vec{x}\right).$$

We ignore the time-dependence, which is not affected by spatial rotations. Now let θ be infinitesimal, and we get

$$\left(1 - \frac{i}{2}\theta\,\hat{n}\cdot\vec{\Sigma}\right)\psi\left(\left(I - \theta\,\hat{n}\cdot\vec{J}\right)\vec{x}\right).$$

But $(I - \theta\hat{n}\cdot\vec{J})\vec{x} = \vec{x} - \theta\,\hat{n}\times\vec{x}$, so the new wave fn is

$$\left(1 - \frac{i}{2}\theta\hat{n}\cdot\vec{\Sigma}\right)\psi\left(\vec{x} - \theta\,\hat{n}\times\vec{x}\right)$$

$$= \psi(\vec{x}) - \theta(\hat{n}\times\vec{x})\cdot\nabla\psi - \frac{i}{2}\theta\,\hat{n}\cdot\vec{\Sigma}\,\psi$$

$$= \psi(\vec{x}) - \frac{i}{\hbar}\theta\,\hat{n}\cdot\left[\vec{x}\times(-i\hbar\nabla\psi) + \frac{\hbar}{2}\vec{\Sigma}\right]\psi$$

But this must be

$$\psi(\vec{x}) - \frac{i}{\hbar}\theta\,\hat{n}\cdot\vec{J}\,\psi(\vec{x})$$

where $\vec{J}$ is the angular momentum (this is the definition of $\vec{J}$).

So,

$$\boxed{\vec{J} = \vec{x}\times\vec{p} + \frac{\hbar}{2}\vec{\Sigma}}$$

where we have used $\vec{p} = -i\hbar\nabla$. The total angular momentum breaks up into an orbital and spin part, that come respectively from rotating the spatial point where the field is evaluated and rotating the spinor. Of course this looks just like the total angular momentum $\vec{J} = \vec{r}\times\vec{p} + \frac{\hbar}{2}\vec{\sigma}$ for the nonrelativistic Pauli theory, except that $\vec{\Sigma}$ is a 4x4 Dirac matrix.

One difference with the NR theory is this. The NR Pauli ~~eqn.~~ Hamiltonian for a free particle commutes with $\vec{L}$ and $\vec{S}$ separately. In the Dirac theory, however, the free particle Hamiltonian does not commute with $\vec{L}$ or $\vec{S} = \frac{\hbar}{2}\vec{\Sigma}$ separately, although it does commute with $\vec{J}$, as it must.

Recall that in the case of the photon (a massless free particle), the operators $\vec{L}$ and $\vec{S}$ do not even exist. $\vec{J}$ however does exist, and single photon states that are eigenstates of J^2, J_z can be constructed.