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$$H = c \vec{\alpha} \cdot \vec{\pi} + mc^2 \beta + q \Phi,$$

$$\text{where } \vec{\pi} = \vec{p} - \frac{q}{c} \vec{A}$$

$\uparrow$  kinetic momentum       $\uparrow$  canonical momentum.

The Heisenberg eqn's are

$$\dot{\mathbf{X}} = \frac{\partial \mathbf{X}}{\partial t} + \frac{1}{i\hbar} [\mathbf{X}, H], \quad \mathbf{X} = \text{any operator.}$$

For  $\mathbf{X} = x_i$  (one of the components of position) we have

$$[x_i, \pi_j] = [x_i, p_j] = i\hbar \delta_{ij}$$

hence

$$\dot{x}_i = \frac{1}{i\hbar} [x_i, H] = \frac{1}{i\hbar} [x_i, c\alpha_j \pi_j] = c\alpha_i,$$

or

$$\boxed{\dot{\vec{x}} = c\vec{\alpha}.}$$

Notice  $[\vec{x}, \alpha] = [\vec{x}, \beta] = 0$  since  $\vec{x}$  is a spatial operator and  $\vec{\alpha}, \beta$  are spin operators. The result is unexpected: the velocity operator  $\dot{\vec{x}}$  is actually a spin operator  $c\vec{\alpha}$ . (The classical relation is relativistic)

$$\dot{\vec{x}} = \frac{\vec{p}/m}{\sqrt{1 + p^2/m^2c^2}}.$$

Notice that ~~the~~ since the eigenvalues of each  $\alpha_k$  are  $\pm 1$ , the eigenvalues of any component of velocity  $v_x = \dot{x}$  for example, are  $\pm c$ . The result of measuring any component of velocity

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of the Dirac electron is  $\pm c$ . Moreover, the different components of velocity don't commute, and so cannot be simultaneously measured.

Now compute the time deriv. of the kinetic momentum  $\vec{\pi}$ .

Use the commutator,

$$[\pi_i, \pi_j] = i\hbar \frac{q}{c} \epsilon_{ijk} B_k.$$

↙ magnetic field.

Note,  $\frac{\partial \pi_i}{\partial t} = -\frac{q}{c} \frac{\partial A_i}{\partial t}$ . So,

$$\dot{\pi}_i = -\frac{q}{c} \frac{\partial A_i}{\partial t} + \frac{q}{c} \epsilon_{ijk} c \alpha_j B_k - q \frac{\partial \Phi}{\partial x_i},$$

or  $\boxed{\dot{\vec{\pi}} = q \left( \vec{E} + \frac{1}{c} \dot{\vec{x}} \times \vec{B} \right)}$ .

This is an operator version of the classical (relativistic) expression for  $\dot{\vec{\pi}}$ , where  $\vec{\pi}$  is the relativistic momentum ( $\gamma m \vec{v}$ ). The formulas are the same, even if the interpretation is different.

Now we look at the nonrelativistic limit of the Dirac equation.

Suppose the energy is  $E = mc^2 +$  small correction of order  $(v/c)^2 \times mc^2$ .

The time dependence of  $\psi$  will be dominated by  $e^{-imc^2 t/\hbar}$ , so

let's strip this off and define a new 4-component spinor with

upper and lower 2-component spinors  $\phi$  and  $\chi$ . That is, put

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$$\begin{pmatrix} \psi \\ \chi \end{pmatrix} = e^{-imc^2 t/\hbar} \begin{pmatrix} \phi \\ \chi \end{pmatrix}.$$

Use the Hamiltonian  $H = c \vec{\alpha} \cdot \vec{\pi} + mc^2 \beta + q\Phi$ . Then

$$\begin{aligned} i\hbar \frac{\partial \psi}{\partial t} &= e^{-imc^2 t/\hbar} \left[ mc^2 \begin{pmatrix} \phi \\ \chi \end{pmatrix} + i\hbar \frac{\partial}{\partial t} \begin{pmatrix} \phi \\ \chi \end{pmatrix} \right] \\ &= e^{-imc^2 t/\hbar} \left[ c \underbrace{\begin{pmatrix} 0 & \vec{\sigma} \cdot \vec{\pi} \\ \vec{\sigma} \cdot \vec{\pi} & 0 \end{pmatrix}}_{\hookrightarrow \equiv \vec{\alpha} \cdot \vec{\pi}} \begin{pmatrix} \phi \\ \chi \end{pmatrix} + mc^2 \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\hookrightarrow \beta} \begin{pmatrix} \phi \\ \chi \end{pmatrix} + q\Phi \begin{pmatrix} \phi \\ \chi \end{pmatrix} \right] \end{aligned}$$

We use the D-P repn of the Dirac matrices. Then we write this out as two coupled eqns connecting two 2-component spinors  $\phi$  and  $\chi$ :

$$mc^2 \phi + i\hbar \frac{\partial \phi}{\partial t} = c (\vec{\sigma} \cdot \vec{\pi}) \chi + mc^2 \phi + q\Phi \phi$$

$$mc^2 \chi + i\hbar \frac{\partial \chi}{\partial t} = c (\vec{\sigma} \cdot \vec{\pi}) \phi - mc^2 \chi + q\Phi \chi$$

or

$$\left. \begin{aligned} i\hbar \frac{\partial \phi}{\partial t} &= c (\vec{\sigma} \cdot \vec{\pi}) \chi + q\Phi \phi \\ 2mc^2 \chi + i\hbar \frac{\partial \chi}{\partial t} &= c (\vec{\sigma} \cdot \vec{\pi}) \phi + q\Phi \chi. \end{aligned} \right\}$$

These are exactly equivalent to the original Dirac eqns. In the 2nd eqn, there are 3 terms multiplying  $\chi$ . The operator

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$i\hbar \frac{\partial \phi}{\partial t}$  acting on  $\chi$  must be regarded as  $\mathcal{O}(v/c)^2$ , since the  $mc^2$  time dependence has already been stripped off, and  $q\Phi$  is of the same order (assuming an essentially NR problem). Thus the term  $2mc^2\chi$  dominates these, and the 2nd eqn is approximately

$$2mc^2\chi = c(\vec{\sigma} \cdot \vec{\pi})\phi$$

or

$$\chi = \frac{1}{2mc} (\vec{\sigma} \cdot \vec{\pi})\phi.$$

Since the kinetic momentum is of order  $\vec{\pi} \sim m\vec{v}$ , we see that

$$\chi \sim \left(\frac{v}{c}\right)\phi,$$

and  $\chi \ll \phi$ . We say,  $\phi$  is the "large" component and  $\chi$  the "small" (this is using the DP repn of the Dirac matrices, for motion at energy  $E = mc^2 + \text{small}$ ). Thus we have solved for  $\chi$  in terms of  $\phi$ . Plugging this back into the  $\phi$  eqn, we get

$$i\hbar \frac{\partial \phi}{\partial t} = \frac{1}{2m} (\vec{\sigma} \cdot \vec{\pi})^2 \phi + q\Phi\phi.$$

Now  $\sigma_i$  and  $\pi_j$  commute ( $\sigma_i$  is purely spin,  $\pi_j$  purely spatial) but  $\sigma_i, \sigma_j$  don't commute, so

$$(\vec{\sigma} \cdot \vec{\pi})^2 = \sigma_i \sigma_j \pi_i \pi_j = (\delta_{ij} + i\epsilon_{ijk} \sigma_k) \pi_i \pi_j = \text{cont'd}$$

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$$= \pi^2 + i \underbrace{\epsilon_{ijk} \sigma_k \pi_i \pi_j}_{\text{crossed out}}$$

$$\rightarrow = \frac{i}{2} \epsilon_{ijk} \sigma_k (\pi_i \pi_j - \pi_j \pi_i)$$

$$= \frac{i}{2} \epsilon_{ijk} \sigma_k \left( i \hbar \frac{q}{c} \right) \epsilon_{ijl} B_l$$

see  $[\pi_i, \pi_j]$  above

$$= - \frac{\hbar q}{2c} \underbrace{\epsilon_{ijk} \epsilon_{ijl}}_{\rightarrow 2\delta_{kl}} \sigma_k B_l = - \frac{\hbar q}{c} \vec{\sigma} \cdot \vec{B}$$

$$\text{So } \frac{1}{2m} (\vec{\sigma} \cdot \vec{\pi})^2 = \frac{1}{2m} \pi^2 - \frac{\hbar q}{2mc} \vec{\sigma} \cdot \vec{B}$$

$$= \frac{1}{2m} \left( \vec{p} - \frac{q}{c} \vec{A} \right)^2 - \underbrace{g \frac{q}{2mc} \vec{S} \cdot \vec{B}}_{\rightarrow -\vec{\mu} \cdot \vec{B}} \quad \left( \vec{S} = \frac{\hbar}{2} \vec{\sigma} \right)$$

$$(g=2)$$

$$\vec{\mu} = g \frac{q}{2mc} \vec{S}$$

The upper equation becomes

$$\boxed{i \hbar \frac{\partial \phi}{\partial t} = \frac{1}{2m} \left( \vec{p} - \frac{q}{c} \vec{A} \right)^2 \phi - \vec{\mu} \cdot \vec{B} \phi + q \Phi \phi}$$

This is the Pauli equation for a spin  $1/2$  particle with charge  $q$  and  $g$ -factor  $g=2$ . We see that the correct Pauli eqn,

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including the interaction of the spin with the magnetic field, including the  $g$ -factor  $g=2$ , emerges from the minimal coupling of the free particle Dirac equ to the EM field.

Next we need to study how the Dirac equ. transforms under Lorentz transformations.

We begin by putting the Dirac equation into covariant notation. It will suffice to do this for the free particle Dirac equation first, which we write in the form

$$i\hbar \frac{\partial \psi}{\partial t} = -i\hbar c \vec{\alpha} \cdot \nabla \psi + mc^2 \beta \psi = H \psi$$

The LHS cannot be a Lorentz scalar, since  $t$  is one component of a 4-vector  $x^\mu = (ct, \vec{x})$ . The Dirac equation as written is specific to one Lorentz frame. Let's bring <sup>all</sup> the derivatives over to the LHS, and write it as

$$i\hbar c \left[ \frac{\partial \psi}{\partial (ct)} + \vec{\alpha} \cdot \nabla \psi \right] = mc^2 \beta \psi.$$

~~The LHS in the square brackets looks like a 4-vector, say  $V^\mu$ , multiplying~~

To put this into covariant form we multiply by  $\beta$  and use  $\beta^2=1$ , to get

$$i\hbar c \left[ \beta \frac{\partial \psi}{\partial x^0} + \beta \vec{\alpha} \cdot \frac{\partial \psi}{\partial \vec{x}} \right] = mc^2 \psi$$

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Now the operator on the RHS,  $mc^2$ , is a Lorentz scalar, so we guess that the coefficients of  $\frac{\partial}{\partial x^\mu}$ , that is  $(\beta, \beta \vec{\alpha})$ , form a 4-vector somehow, and we define

$$\gamma^\mu = (\gamma^0, \vec{\gamma}) = (\beta, \beta \vec{\alpha}),$$

that is 
$$\left. \begin{aligned} \gamma^0 &= \beta \\ \vec{\gamma} &= \beta \vec{\alpha} \end{aligned} \right\}.$$

Then the Dirac eqn. becomes

$$i\hbar \gamma^\mu \frac{\partial \psi}{\partial x^\mu} = mc \psi,$$

cancelling a factor of  $c$ . If we introduce the covariant momentum operators,

$$p_\mu = i\hbar \partial_\mu = i\hbar \frac{\partial}{\partial x^\mu},$$

then we have

$$(\gamma^\mu p_\mu - mc) \psi = 0$$

for the free particle Dirac eqn. ~~which is~~ The notation is suggestive, but we have not shown yet in what sense  $\gamma^\mu$  forms a 4-vector. We will do that later; for now just notice that  $\gamma^\mu$  is a 4-component vector of  $4 \times 4$  Dirac matrices (like  $\vec{\sigma}$  is a 3-vector of  $2 \times 2$  matrices).

We can introduce the coupling with the EM field by means of the

minimum coupling prescription,  $p_\mu \rightarrow p_\mu - q/c A_\mu$ , where  $q$  is the charge of the particle and

$$A^\mu = (\Phi, \vec{A})$$

$$A_\mu = (\Phi, -\vec{A}).$$

Then the Dirac eqn. becomes

$$(\gamma^\mu p_\mu - \frac{q}{c} \gamma^\mu A_\mu - mc) \psi = 0.$$

Contractions between  $\gamma^\mu$  and an ordinary 4-vector like  $A_\mu$ , or a 4-vector of operators like  $p_\mu$ , are very common. Here is some notation for such contractions. Let  $X^\mu$  be any 4-vector. Then

$$\not{X} = \gamma^\mu X_\mu$$

is called the Feynman slash. In terms of this notation, the Dirac equation becomes

$$\boxed{(\not{p} - \frac{q}{c} \not{A} - mc) \psi = 0.}$$

This is regarded as the covariant version of the Dirac equation. It is equivalent to the Hamiltonian version  $i\hbar \frac{\partial \psi}{\partial t} = H \psi$ ,  $H =$  Dirac Hamiltonian, quoted above.

The covariant form of the Dirac equation is equivalent (in the NR limit) to the Pauli equation with  $g=2$ , as we have just shown. And yet it is simpler than the Pauli equation in form, in spite of the



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extra notation. In fact, it contains even more physics than the Pauli equation, as we shall show later, when it is expanded to higher order in  $(v/c)^2$ .

The  $\gamma^\mu$  matrices are an alternative form of the Dirac matrices, useful when we wish to reveal the covariant aspects of the Dirac equation. All the properties of the  $\vec{\alpha}, \beta$  matrices can be converted into properties of the  $\gamma^\mu$  matrices. Here we list some of them.

First, there are the values of these matrices. We have

$$\gamma^0 = \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (\text{DP}) \quad \text{or} \quad \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \quad (\text{Weyl}).$$

$$\vec{\gamma} = \beta \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad (\text{DP}) \quad \text{or} \quad \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \quad (\text{Weyl})$$

Next, the Hermiticity properties. Recall that  $\vec{\alpha}, \beta$  are Hermitian.

This implies

$$\gamma^0 = \beta = \text{Hermitian}$$

$$\gamma^i = \beta \alpha_i = \text{anti-Hermitian.}$$

For example,

$$\begin{aligned} (\gamma^i)^\dagger &= (\beta \alpha_i)^\dagger = \alpha_i \beta = -\beta \alpha_i \quad \text{since } \{\beta, \alpha_i\} = 0 \\ &= -\gamma^i. \end{aligned}$$

Remark. Why do we use an upper (contravariant) index on  $\gamma^i$  but a lower  $\mathbb{F}$  index on  $\alpha_i$ ? Because  $\gamma^i$  is the spatial part

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of a 4-vector  $\gamma^\mu$ , and it is necessary to indicate whether a contravariant or covariant index is intended. As for  $\alpha_i$ , it is never used as the spatial part of a 4-vector (there is no " $\alpha^0$ ") so we just use a lower index in all cases. The same is true of the velocity  $\vec{v} = d\vec{x}/dt$ , which is never the spatial part of a 4-vector.

Third, the anti commutation relations constituting the Dirac algebra can be ~~put into~~ expressed in terms of the  $\gamma^\mu$  matrices. Note:

$$\{\gamma^0, \gamma^0\} = \{\beta, \beta\} = 2$$

$$\{\gamma^0, \gamma^i\} = \{\beta, \beta\alpha_i\} = \beta\beta\alpha_i + \overleftarrow{\beta\alpha_i}\beta = \beta^2\alpha_i - \beta^2\alpha_i = 0.$$

$$\begin{aligned} \{\gamma^i, \gamma^j\} &= \{\beta\alpha_i, \beta\alpha_j\} = \beta\overleftarrow{\alpha_i}\beta\alpha_j + \overleftarrow{\beta\alpha_j}\beta\alpha_i = -\beta^2\alpha_i\alpha_j - \beta^2\alpha_j\alpha_i \\ &= -\{\alpha_i, \alpha_j\} = -2\delta_{ij}. \end{aligned}$$

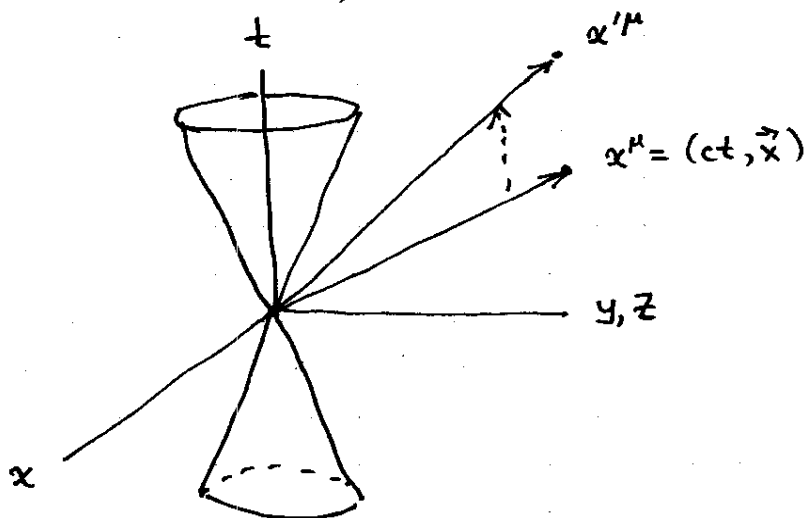
In summary,

$$\boxed{\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}}$$

This is the most compact form of the Dirac algebra; it looks covariant, and indeed we will see that it is so, once we have worked out the transformation properties of the  $\gamma^\mu$  matrices under Lorentz transformations.

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We put the Dirac equation on hold for the time being and consider Lorentz transformations. We make a sketch of space-time with the time axis vertical,



A point of space-time is called an "event".

The light cone is shown (the locus of events generated by a light flash at  $t=x=y=z=0$ ). One spatial dimension is suppressed to make a ~~plot~~ <sup>sketch</sup> possible. One event with coordinates  $x^\mu$  is shown. We consider a mapping of space-time onto itself, taking  $x^\mu$  into  $x'^\mu$  as shown. We suppose the mapping is linear and specified by a matrix  $\Lambda^\mu{}_\nu$ ,

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

The little dot in  $\Lambda^\mu{}_\nu$  is a place holder, to indicate that the first index is upper or contravariant.

We define the mapping to be a Lorentz transformation if the Minkowski scalar product is preserved, i.e. if

$$x'^\mu g_{\mu\nu} x'^\nu = x^\alpha g_{\alpha\beta} x^\beta \quad \text{for all initial } x^\alpha.$$

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Notice that this means that the nature of the vector  $x^\mu$  (space-like, time-like or light-like) is preserved by a Lorentz transformation. In particular, the outside of the light cone is mapped into itself, the inside is mapped into itself, and the light cone is mapped into itself. As we shall see, the interior of the forward light cone may be mapped into itself, or into the interior of the backward light cone (it depends on the Lorentz transformation).

Substituting  $x'^\mu = \Lambda^\mu{}_\nu x^\nu$  into the eqn. above, we get

$$x^\alpha \Lambda^\mu{}_\alpha g_{\mu\nu} \Lambda^\nu{}_\beta x^\beta = x^\alpha g_{\alpha\beta} x^\beta \quad \text{for all } x^\alpha,$$

or

$$\boxed{\Lambda^\mu{}_\alpha g_{\mu\nu} \Lambda^\nu{}_\beta = g_{\alpha\beta}}$$

This is the equation that  $\Lambda^\mu{}_\nu$  must satisfy to be a Lorentz transformation.

Let's put this into matrix form. We define

$\Lambda =$  matrix with components  $\Lambda^\mu{}_\nu$  ← row

$g =$  " " " ← column

$g^{-1} =$  " " " ← column

i.e.  $\begin{pmatrix} +1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Actually,  $g$  and  $g^{-1}$  are the same matrices, but we'll keep them notationally distinct anyway. Then the defining condition of a Lorentz transformation is

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$$\Lambda^t g \Lambda = g$$

You should compare this with the definition of an <sup>3x3</sup> orthogonal matrix,

$$R^t I R = I \quad \Rightarrow \quad R \in O(3)$$

where  $I$  is the  $3 \times 3$  identity. The set of all  $3 \times 3$  orthogonal matrices forms a group, denoted  $O(3)$ . By comparison, we see that a Lorentz transformation is a kind of orthogonal matrix with respect to the Minkowski metric  $g_{\mu\nu}$  instead of the Euclidean metric  $I$ . It is easy to show that the set of Lorentz transformations by our definition forms a group; this group is denoted  $O(3,1)$ , meaning the set of matrices orthogonal w.r.t. a metric with 3 space-like and 1 time-like directions.

The proof of the group property works like this. First note that  $\Lambda = I$  is a Lorentz transformation (the <sup>4x4</sup> identity), since

$$I^t g I = g.$$

Next, by taking determinants, we find

$$(\det \Lambda)^2 = 1 \quad (\text{the } \det g \text{ cancels from both sides})$$

so  $\det \Lambda = \pm 1$ . Thus  $\Lambda^{-1}$  always exists. In fact by multiplying by  $g^{-1}$  we get

$$(g^{-1} \Lambda^t g) \Lambda = I,$$

or

$$\Lambda^{-1} = g^{-1} \Lambda^t g.$$

This is not quite as simple as the rule  $R^{-1} = R^t$  for ordinary orthogonal matrices, but it does make it easy to invert a Lorentz transformation. In components this relation is

$$(\Lambda^{-1})^\mu{}_\nu = g^{\mu\alpha} \Lambda^\beta{}_\alpha g_{\beta\nu}.$$

In any case, take  $\Lambda^t g \Lambda = g$  and multiply from the left by  $\Lambda^{t-1}$  and from the right by  $\Lambda^{-1}$ , and we get

$$g = (\Lambda^{-1})^t g (\Lambda^{-1})$$

so  $\Lambda^{-1}$  is also a Lorentz transformation. Finally, if  ~~$\Lambda_1$~~   $\Lambda_1$  and  $\Lambda_2$  are Lorentz transformations and  $\Lambda = \Lambda_1 \Lambda_2$ , then

$$\Lambda^t g \Lambda = \Lambda_2^t \underbrace{\Lambda_1^t g \Lambda_1}_g \Lambda_2 = \Lambda_2^t g \Lambda_2 = g,$$

so  $\Lambda$  is also.

We are viewing Lorentz transformations as mappings of space-time onto itself, that is, we are taking the active point of view. Most introductions to relativity theory take the passive point of view, in which coordinates or tensors are transformed from one coordinate system to another. But we used the active point of view in the theory of rotations in quantum mechanics last semester, and

we will continue it with Lorentz transformations. We'll give some examples presently of how active Lorentz transformations are used in practice.

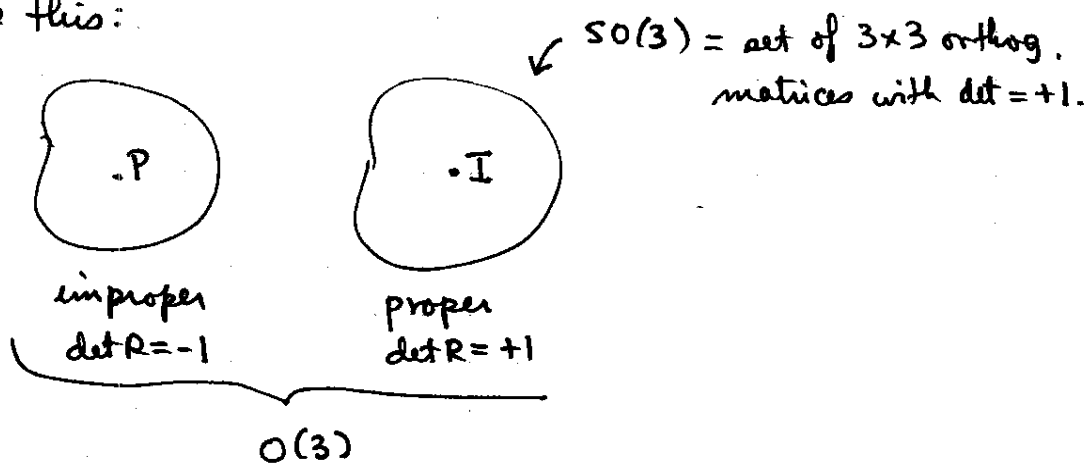
The space of Lorentz transformations is 6-dimensional, that is, it takes 6 parameters to specify a L.T. Physically these parameters can be identified with the 3 Euler angles of a spatial rotation, plus the 3 components of a velocity  $\vec{v}$  used in a boost. The fact that 6 parameters are required also follows from the definition  $\Lambda^t g \Lambda = g$ . An arbitrary, real  $4 \times 4$  matrix requires  $4^2 = 16$  numbers for its specification, but  $\Lambda^t g \Lambda = g$  is a  $4 \times 4$  symmetric matrix of constraints that  $\Lambda$  must satisfy to be a L.T. A symmetric  $4 \times 4$  matrix has 10 independent components, so there are 16 parameters - 10 constraints = 6 parameters left over to specify a L.T. The group manifold of  $O(3,1)$  can be thought of as the 6-dimensional surface in the 16-dimensional space of  $4 \times 4$  matrices, where matrices on the surface satisfy  $\Lambda^t g \Lambda = g$ .

Last semester we analyzed the similar count of parameters for the group  $O(3)$  of  $3 \times 3$  orthogonal matrices. In that case, matrix space is 9-dimensional, and the condition  $R^t R = I$  provides 6 constraints, so an orthogonal matrix  $R \in O(3)$

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requires  $9-6=3$  parameters for its specification (for example, the 3 Euler angles). Equivalently, the group manifold of  $O(3)$  is a 3-dimensional surface in 9-d matrix space.

As we saw last semester, the group manifold  $O(3)$  consists of two disconnected components (each component is connected, but disconnected from each other). The matrices in the two components are distinguished by  $\det R = \pm 1$ . Those with  $\det R = +1$  are proper rotations and those with  $\det R = -1$  are improper. So the group manifold  $O(3)$  looks something like this:



The proper component is the group manifold  $SO(3)$ . It contains the identity  $I$ , and is called the identity component. Any proper rotation can be continuously connected with the identity. The improper component contains the spatial inversion matrix

$$P = \begin{pmatrix} -1 & & 0 \\ & -1 & \\ 0 & & -1 \end{pmatrix} = -I.$$



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Since the Lorentz group  $O(3,1)$  contains matrices with  $\det \Lambda = \pm 1$ , we expect that the Lorentz group manifold <sup>also</sup> consists of at least 2 disconnected components. This manifold is a 6-d surface in 16-d matrix space. Actually, the manifold  $O(3,1)$  consists of 4 disconnected components. The choice is made by  $\det \Lambda$ , which can be  $\pm 1$ , and the sign of  $\Lambda^0_0$  (the 0-0 component of  $\Lambda$ ).

The meaning of  $\Lambda^0_0$  is that it is the coefficient connecting the old time and the new time, i.e., in  $x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu}$ , we have

$$t' = \Lambda^0_0 t + \text{other terms}$$

$$\text{or } \Lambda^0_0 = \frac{\partial t'}{\partial t}$$

It is easy to show from the definition  $\Lambda^t \eta \Lambda = \eta$  that  $|\Lambda^0_0| \geq 1$ . If  $\Lambda^0_0$  is positive (hence  $\Lambda^0_0 \geq 1$ ) it means that when the old time  $t$  increases, then so does  $t'$ . In that case,  $\Lambda^0_0 = \partial t' / \partial t = \gamma = 1/\sqrt{1-v^2/c^2}$  where  $v$  is the velocity of the boost contained in  $\Lambda$ . It is the usual relativistic factor of time dilation. When  $\Lambda^0_0$  is negative (hence  $\Lambda^0_0 \leq -1$ ), then when  $t$  increases, the new time  $t'$  <sup>old time</sup> decreases. This means that the L.T.  $\Lambda$  contains a time-reversal operation, perhaps in addition to rotations and boosts. Such a L.T. maps the forward light cone into the backward light cone.

Here are some examples of ~~the~~ matrices taken from the four

components of  $O(3,1)$ :

matrix	$\det \Lambda$	sign $\Lambda^0_0$
I	+1	+1
P	-1	+1
T	-1	-1
PT	+1	-1

where

$$I = \begin{pmatrix} +1 & & & \\ & +1 & & \\ & & 0 & \\ & & & +1 \end{pmatrix}$$

is the identity, where

$$P = \begin{pmatrix} +1 & & & \\ & -1 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}$$

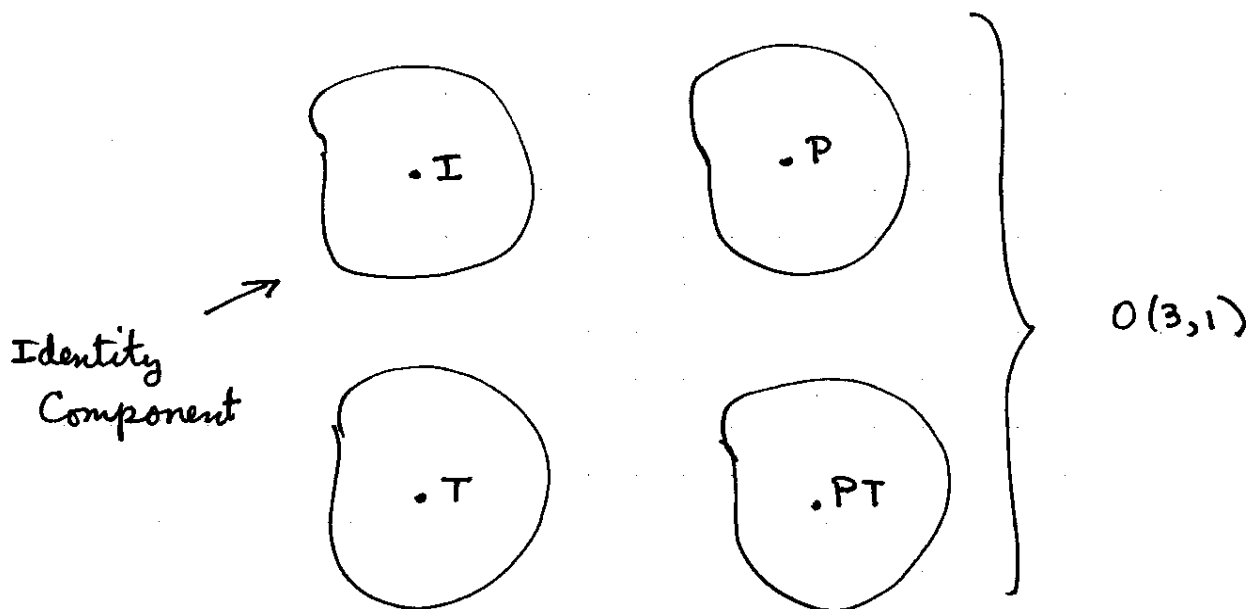
is the L.T. that reverses all spatial vectors (it is the spatial inversion operation, related to parity); where

$$T = \begin{pmatrix} -1 & & & \\ & +1 & & \\ & & 0 & \\ & & & +1 \end{pmatrix}$$

is the operation that reverses the direction of time, and where

$$PT = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & 0 & \\ & & & -1 \end{pmatrix}.$$

Thus, the group manifold  $O(3,1)$  looks something like this:



If  $\Lambda$  belongs to the identity component, then we say that it is a proper Lorentz transformation. For the time being, we will restrict attention only to the proper Lorentz transformation, but we will return to parity somewhat later. We probably won't have time in this course for time-reversal in relativistic systems.

A useful theorem is that every proper Lorentz transformation can be uniquely factored into a rotation times a boost:

$$\Lambda = R B$$

The rotations are just the same as the rotations we studied last semester, except now represented by  $4 \times 4$  matrices instead of  $3 \times 3$ , with an extra row and column added to take care of the time component, which is not changed by a rotation. For example,

the rotations about the 3 coordinate axes are

$$R(\hat{x}, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos \theta & -\sin \theta \\ 0 & 0 & \sin \theta & \cos \theta \end{pmatrix}$$

$$R(\hat{y}, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & 0 & \sin \theta \\ 0 & 0 & 1 & 0 \\ 0 & -\sin \theta & 0 & \cos \theta \end{pmatrix}$$

$$R(\hat{z}, \theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta & 0 \\ 0 & \sin \theta & \cos \theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Boosts are defined in the usual way in relativity theory, except that we are taking the active point of view. If  $v$  is the velocity of a boost, then we define the usual (Jackson) <sup>ala</sup> parameters  $\beta, \gamma$  by

$$\beta = \frac{v}{c},$$

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}} = \frac{1}{\sqrt{1 - \beta^2}}.$$

of course these are not to be confused with Dirac matrices.

Boosts are conveniently parameterized by the rapidity  $\lambda$ , defined by

$$\begin{aligned}\beta &= \tanh \lambda \\ \gamma &= \cosh \lambda \\ \gamma\beta &= \sinh \lambda.\end{aligned}$$

Note that since  $-1 < \beta < 1$ ,  $-\infty < \lambda < +\infty$ . In terms of  $\lambda$ , the boosts along the 3 coordinate axes are

$$B(\hat{x}, \lambda) = \begin{pmatrix} \cosh \lambda & \sinh \lambda & 0 & 0 \\ \sinh \lambda & \cosh \lambda & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B(\hat{y}, \lambda) = \begin{pmatrix} \cosh \lambda & 0 & \sinh \lambda & 0 \\ 0 & 1 & 0 & 0 \\ \sinh \lambda & 0 & \cosh \lambda & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$B(\hat{z}, \lambda) = \begin{pmatrix} \cosh \lambda & 0 & 0 & \sinh \lambda \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh \lambda & 0 & 0 & \cosh \lambda \end{pmatrix}$$

Let us illustrate a boost, to show that it is correct in the active sense. Let  $p_0^\mu$  be the 4-momentum of a particle at rest:

$$p_0^\mu = \begin{pmatrix} mc \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Now apply a boost along the  $x$ -axis, using  $\cosh\lambda = \gamma$ ,  $\sinh\lambda = \gamma\beta$ :

$$p^\mu = B(\hat{x}, \lambda)^\mu{}_\nu p_0^\nu = \begin{pmatrix} mc\gamma \\ mc\gamma\beta \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E/c \\ \vec{p} \end{pmatrix},$$

where  $E = mc^2\gamma$  and  $\vec{p} = \gamma m\vec{v}$ ,  $\vec{v} = v\hat{x}$ . These are the correct relativistic ~~relations~~ <sup>expressions</sup> for the energy and momentum of a particle moving with velocity  $\vec{v}$  down the  $x$ -axis.