In general, measurements in QM are subject to fluctuations, as we make them across an ensemble of identically prepared systems. The same must be true for the quantum fields $\vec{A}(\vec{x})$, $\vec{E}(\vec{x})$, $\vec{B}(\vec{x})$, etc. This is in contrast to classical electromagnetism, in which one imagines that field strengths can be measured with arbitrary precision and produce a definite answer. We now examine this question, working with the free field for simplicity.

We start with the electric field, given by

$$\vec{E}(\vec{x}) = \sqrt{2\pi\hbar} \int \frac{d^3\vec{k}}{(2\pi)^{3/2}} \sum_{\mu} \sqrt{\omega_{\vec{k}}} \left[i\, a_\mu(\vec{k})\, \hat{\varepsilon}_\mu(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} + h.c. \right].$$

Notice that $\vec{x}$ here is the location at which the field is measured; it is not an operator. The operator is $\vec{E}$, corresponding to what is measured. If we measure $\vec{E}$ at a specific point $\vec{x}$ when the quantum state of the field is the vacuum $|0\rangle$, then the average value obtained is

$$\langle 0|\vec{E}(\vec{x})|0\rangle = 0 .$$

This follows because $\vec{E}(\vec{x})$ is a linear combination of creation and annihilation operators, $a^\dagger_\mu(\vec{k})$ and $a_\mu(\vec{k})$, and

$$a_\mu(\vec{k})|0\rangle = 0 \qquad \text{and} \qquad \langle 0| a^\dagger_\mu(\vec{k}) = 0 .$$

This makes sense: the average value of $\vec{E}(\vec{x})$ in the vacuum should be zero.

But this doesn't mean that individual measurements will give zero, only the average. To see what happens to individual measurements, let us compute the dispersion, which is $\langle 0|\vec{E}(\vec{x})^2|0\rangle$. This is

$$\langle 0|\vec{E}(\vec{x})^2|0\rangle = 2\pi\hbar \int \frac{d^3\vec{k}\, d^3\vec{k}'}{(2\pi)^3} \sum_{\mu\mu'} \sqrt{\omega\omega'}$$

$$\langle 0| \left[i a_\mu(\vec{k})\hat{\varepsilon}_\mu(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} - i a_\mu^\dagger(\vec{k})\hat{\varepsilon}_\mu^*(\vec{k})\, e^{-i\vec{k}\cdot\vec{x}} \right]$$

$$\cdot \left[i a_{\mu'}(\vec{k}')\hat{\varepsilon}_{\mu'}(\vec{k}')\, e^{i\vec{k}'\cdot\vec{x}} - i a_{\mu'}^\dagger(\vec{k}')\hat{\varepsilon}_{\mu'}^*(\vec{k}')\, e^{-i\vec{k}'\cdot\vec{x}} \right] |0\rangle.$$

Of the four major terms, only the one involving $a_\mu(\vec{k})\, a_{\mu'}^\dagger(\vec{k}')$ is nonzero, because all the others have an annihilation operator adjacent to the vacuum ket $|0\rangle$ or a creation operator adjacent to the vacuum bra $\langle 0|$. As for the one term that is nonzero, we have

$$\langle 0|\, a_\mu(\vec{k})\, a_{\mu'}^\dagger(\vec{k}')|0\rangle$$

$$= \langle 0|\, a_{\mu'}^\dagger(\vec{k}')\, a_\mu(\vec{k}) + \delta_{\mu\mu'}\,\delta(\vec{k}-\vec{k}')\,|0\rangle = \delta_{\mu\mu'}\,\delta(\vec{k}-\vec{k}'),$$

where we use the commutator of a's and $a^\dagger$'s. The Kronecker-Dirac deltas allow us to do the μ' sum and the $\vec{k}'$ integral. The dot product simplifies,

$$\hat{\varepsilon}_\mu(\vec{k})\cdot\hat{\varepsilon}_{\mu'}^*(\vec{k}') \longrightarrow \hat{\varepsilon}_\mu(\vec{k})\cdot\hat{\varepsilon}_\mu^*(\vec{k}) = 1,$$

~~so~~ and the phase factors disapper, $e^{i\vec{k}\cdot\vec{x} - i\vec{k}'\cdot\vec{x}} \to 1$, so

$$\langle 0 | E^2(\vec{x}) | 0 \rangle = 2\pi\hbar \int \frac{d^3\vec{k}}{(2\pi)^3} \sum_\mu \omega$$

$$= \frac{(2\pi\hbar c)(2)}{(2\pi)^3} \underbrace{\int d^3\vec{k}\; k}_{\textstyle \longrightarrow\ 4\pi \int_0^\infty k^2 dk\; k}\,,$$

where $\sum_\mu \to 2$ since the integrand is independent of μ, and the 4π is the integral over solid angles in $\vec{k}$-space. The final integral diverges, but if we replace the upper limit by $k = K$, where K is a cut-off, then we get

$$\langle 0 | E^2(\vec{x}) | 0 \rangle = \frac{\hbar c}{2\pi} K^4 . \to \infty \quad \text{as } K \to \infty .$$

We should not be surprised by the infinite result, since $E^2/8\pi$ is one term in the energy density of the field, and the vacuum zero-point energy density is infinite. Although we threw away the zero-point energy in the Hamiltonian, it reappears in the computation of the dispersion in the measured value of the electric field strength.

How do we reconcile this result with the fact that real measurements of $\vec{E}$ always give a finite value? One way to understand this is to note that real measuring devices occupy a finite volume, and hence measure the average of $\vec{E}$ over some region.

Let $\mathcal{R}$ be a region of space with volume $\mathcal{V}$ (not to be confused with the volume of the box — we are not using box normalization here anyway), and let us define the average electric field,

$$\bar{\vec{E}} = \frac{1}{\mathcal{V}} \int_{\mathcal{R}} d^3\vec{x} \;\; \vec{E}(\vec{x}).$$

Then we can easily see that $\langle 0 | \bar{\vec{E}} | 0 \rangle = 0$. As for the dispersion, it is

$$\langle 0 | \bar{\vec{E}}^2 | 0 \rangle = \frac{1}{\mathcal{V}^2} \int_{\mathcal{R}} d^3\vec{x} \int_{\mathcal{R}} d^3\vec{x}' \; (2\pi\hbar) \int \frac{d^3\vec{k}\, d^3\vec{k}'}{(2\pi)^3} \sqrt{\omega\omega'} \sum_{\mu\mu'}$$

$$\langle 0 | \left[i\, a_\mu(\vec{k})\, \hat{\varepsilon}_\mu(\vec{k})\, e^{i\vec{k}\cdot\vec{x}} + h.c. \right]$$

$$\cdot \left[\quad h.c. \quad - i\, a^\dagger_{\mu'}(\vec{k})\, \hat{\varepsilon}^*_{\mu'}(\vec{k}')\, e^{-i\vec{k}'\cdot\vec{x}'} \right] | 0 \rangle$$

Again only the $a_\mu(\vec{k})\, a^\dagger_{\mu'}(\vec{k}')$ - term survives, and this simplifies to

$$\frac{1}{\mathcal{V}^2} \int_{\mathcal{R}} d^3\vec{x} \int_{\mathcal{R}} d^3\vec{x}' \;\; (2\pi\hbar) \int \frac{d^3\vec{k}}{(2\pi)^3} \; \underbrace{(2)}_{\text{from } \sum_\mu} \; \omega \;\; e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} .$$

We will estimate this integral as an order of magnitude. Since both $\vec{x}, \vec{x}' \in \mathcal{R}$, the distance $|\vec{x}-\vec{x}'|$ is $<$ the linear dimension of $\mathcal{R}$, call it L, so that $\mathcal{V} \sim L^3$.

Thus if $k \ll \frac{1}{L}$, then $\vec{k}\cdot(\vec{x}-\vec{x}') \ll 1$, and

$$e^{i\vec{k}\cdot(\vec{x}-\vec{x}')} \approx 1.$$

But if $k \gg 1/L$, then $e^{i\vec{k}\cdot(\vec{x}-\vec{x}')}$ is rapidly oscillating, and it chops up the rest of the integrand to give effectively zero. This means we can estimate the value of the integral by setting the upper limit on k to $K = 1/L$. Then the $\vec{k}$ integral can be done, and it gives the same result obtained previously, with the cutoff of K. The result is independent of $\vec{x}$ and $\vec{x}'$, so

$$\frac{1}{v}\int d^3\vec{x} \to 1 \quad \text{and} \quad \frac{1}{v}\int d^3\vec{x}' \to 1,$$

and we get, as an order of magnitude,

$$\langle 0|\vec{E}^2|0\rangle = \hbar c K^4 = \frac{\hbar c K}{v} = \frac{\hbar\omega}{v},$$

where ω is the cutoff frequency cK.

Quantum electrodynamics predicts that the measurement of electric field strength, with an instrument occupying a volume $v = L^3$, will produce fluctuations whose square is given by $\hbar\omega/v$, $\omega = cK = c/L$. This is in the vacuum (in the absence of any applied field).

Are these fluctuations real? Yes, charged particles respond to them, and they modify the dynamics of the particle. For example, the <u>Lamb shift</u> is due to the interaction of the atomic electron with the fluctuating electromagnetic field.

Suppose we are interested in detecting an electromagnetic signal, e.g. a radio transmission. Using classical EM theory, we calculate an electric field E_{signal} at the location of our detector. But the detector will also pick up quantum fluctuations, call them E_{quant}. The classical signal will only be detected cleanly if $E_{signal} \gg E_{quant}$.

Let the size of the antenna be λ, the wavelength. ~~Measure~~ Measure E_{signal} by the energy density, expressed in terms of the number of photons per unit volume, n. Then

$$E_{signal}^2 \sim n\hbar\omega \gg E_{quant}^2 \sim \frac{\hbar\omega}{\lambda^3}.$$

This gives

$$n\lambda^3 \gg 1,$$

the number of photons in a cubic wavelength must be $\gg 1$. This is the usual criterion for the validity of classical E+M.