Physics 221B
Spring 2012
Homework 24
Due Friday, April 13, 2012

Reading Assignment:

This week we finished the material on free particle solutions of the Dirac equation, in which I followed Bjorken and Drell up through about p. 33, where they start to talk about projection operators for spin (which we didn't cover). The rest of the week was spent covering various topics that amount to exploring the properties of the Dirac equation, finding out what works and what gives difficulties of interpretation. These topics included: the Gordon decomposition of the current; the Heisenberg equations of motion for the free particle and their use in understanding the Zitterbewegung; the Klein paradox; the hydrogen atom; and the Foldy-Wouthuysen transformation. The material on these topics is spread among handwritten notes, Bjorken and Drell and Sakurai.

The Gordon decomposition of the current is discussed in Bjorken and Drell, pp. 36–37, but my handwritten notes are more clear in terms of what the point is and the derivation itself. The point is to try to understand the strange fact that the velocity operator in the Dirac theory, $\mathbf{v} = \dot{\mathbf{x}} = c\boldsymbol{\alpha}$, is a purely spinor operator. The same operator appears in the spatial part of the probability current, $\mathbf{J} = c\psi^\dagger \boldsymbol{\alpha}\psi$. One of the conclusions is that the probability current should contain a spin contribution, and it does so even in the nonrelativistic theory of a particle of nonzero spin (in the magnetization term).

I followed Sakurai, pp. 112–113 and 115–117, in deriving and solving the Heisenberg equations of motion for the velocity for a free particle. Unfortunately there are no handwritten notes on this material. The point of doing this is that by taking expectation values of the solution of the Heisenberg equations of motion with respect to a wave packet we obtain the time-dependence of the expectation value of the velocity operator, which, as strange as it may seem, is not a constant of the motion in the Dirac theory of the free particle (although the momentum and energy are). One then finds that the time scale $\tau \sim \hbar/mc^2$ plays a role; this is the time scale associated with the rest mass of the particle. For times much less than this time, the wave packet moves on a straight line with a constant velocity; in an eigenfunction of one component of the velocity operator, this is the speed of light. For much longer times, the wave packet moves with an average velocity given in terms of the momentum and energy by $\mathbf{v} = c^2\mathbf{p}/E$, which is the formula from classical relativity theory connecting the velocity with the momentum. The intuitive idea is that the particle travels at the speed of light for a time $\hbar/mc^2$, covering a distance of $\hbar/mc$ (the Compton wave length); then it changes direction randomly. Over times long compared to $\hbar/mc^2$ the particle effectively gets smeared out over a distance of order of the Compton wavelength, while undergoing a slower drift of velocity $c^2\mathbf{p}/E$. Both Bjorken and Drell and Sakurai explore these properties by explicitly constructing a wave packet as a linear combination of free particle solutions (pp. 36–38 of B&D, pp. 117–118 of Sakurai), and then looking at the time evolution; in my opinion these calculations are messy and it's easier just to sandwich the solution of the Heisenberg equations of motion with a wave packet, as I argued in class. In any case, the point is the Zitterbewegung.

The Klein paradox is discussed in Bjorken and Drell, pp. 40–42; I just presented the main results in class (when the barrier height exceeds $2mc^2$, you start to get a reflected wave with more flux than the incident flux, and a transmitted wave with a backwards flux). The point is that this is a paradox of the Dirac equation that has no easy resolution in the context that we are exploring, except to note that it is related to the negative energy solutions for which we have no interpretation.

The hydrogen atom in the Dirac equation is discussed by Bjorken and Drell, pp. 52–56. In class I just presented the main result (the energy eigenvalues), noting that when expanded in powers of $Z\alpha$ they agree with the fine structure correction in hydrogen.

The FW transformation is discussed by B&D, pp. 46–52; my handwritten notes follow this pretty closely, with some elaboration and clarifications. The main point of this is that the Dirac equation can be expanded in powers of v/c, and, when carried through fourth order, it gives all three of the fine structure corrections that we explored last semester. One can only conclude that the Dirac equation, in spite of the paradoxes and difficulties of interpretation, must have a great deal of truth in it.

1. This is Bjorken and Drell problem 4.2, with the steps laid out in more detail. Do this problem in the following way. First, use natural units, $\hbar = c = 1$. Next, take the modified Dirac equation to be

$$\Big(\not{p} - q\not{A} - \frac{\kappa e}{4m}\sigma_{\mu\nu}F^{\mu\nu} - m\Big)\psi = 0, \tag{1}$$

where m is the mass, q the charge, and κ the strength of the anomalous magnetic moment term. For the electron, $q = -e$ and $\kappa = 0$; for the proton, $q = e$ and $\kappa = 1.79$; and for the neutron, $q = 0$ and $\kappa = -1.91$. Beware, Bjorken and Drell define $F_{\mu\nu}$ with an opposite sign compared to Jackson; I think Jackson's formula,

$$F_{\mu\nu} = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu}, \tag{2}$$

is more standard nowadays, and you should follow it.

(a) Write out the modified Dirac Hamiltonian, and show that it is Hermitian.

(b) Show that probability is conserved, i.e.,

$$\frac{\partial J^\mu}{\partial x^\mu} = 0, \tag{3}$$

where J^μ is defined exactly as for the unmodified Dirac equation, $J^\mu = \bar{\psi}\gamma^\mu\psi$.

(c) Covariance. Suppose $\psi(x)$ satisfies the modified Dirac equation (1), and let

$$\begin{aligned} \psi'(x) &= D(\Lambda)\psi(\Lambda^{-1}x), \\ A'^\mu(x) &= \Lambda^\mu{}_\nu\, A^\nu(\Lambda^{-1}x), \\ F'^{\mu\nu}(x) &= \Lambda^\mu{}_\alpha\, \Lambda^\nu{}_\beta\, F^{\alpha\beta}(\Lambda^{-1}x). \end{aligned} \tag{4}$$

Then show that $\psi'(x)$ satisfies the modified Dirac equation (1), but with Lorentz transformed fields $A'^\mu(x)$ and $F'^{\mu\nu}(x)$ instead of the original fields.

(d) Assume $\mathbf{E} = 0$, $\mathbf{B} \neq 0$ (in order to see what the effective magnetic moment of the particle is). Perform a simple nonrelativistic approximation as in pp. 2–6 of the lecture notes for March 9, and show that you get the right g-factors for the proton and neutron.

(e) Carry out a systematic Foldy-Wouthuysen transformation for the neutron as requested by problem 4.2. Remember $q = 0$, which simplifies the calculation. Order the terms in powers of $v/c = \eta$, as done in class, and carry the expansion out to order η^4.