Physics 221B
Spring 2012
Homework 23
Due Friday, April 6, 2012

Reading Assignment:

Handwritten notes for the week on the Dirac equation. These notes follow Bjorken and Drell pretty closely. See Bjorken and Drell, Chapter 2, pp. 22–26, and Chapter 3, pp. 28–33, but you can skip the material on projection operators for spin. See also Sakurai, pp. 99–107.

1. Show that the quantity $\bar{\psi}(x)\gamma_5\gamma^\mu\psi(x)$ transforms as a pseudovector under Lorentz transformations.

2. This is a version of Bjorken and Drell problem 1, p. 42. We will use the Hermitian orthonormality relation (3.11) more than the adjoint version (3.9b). It may be proved just by brute force multiplication of the columns of the matrix $D(\Lambda)$, as suggested by Bjorken and Drell, but a representation-independent proof is preferable.

(a) Let $u(\mathbf{0})$ be any linear combination of the spinors $w^1(\mathbf{0})$ and $w^2(\mathbf{0})$, that is, let it be the spinor part of a positive energy solution of the free particle Dirac equation for a particle at rest. Similarly, let $v(\mathbf{0})$ be any linear combination of $w^3(\mathbf{0})$ and $w^4(\mathbf{0})$. Let $\mathbf{p}$ be a momentum and let

$$E = E(\mathbf{p}) = \sqrt{m^2c^4 + c^2p^2}. \tag{1}$$

Let $\hat{\mathbf{b}} = \mathbf{p}/p$ and let λ be a rapidity defined by $\cosh\lambda = E/mc^2$, $\sinh\lambda = p/mc$. Let $\Lambda(\mathbf{p}) = \Lambda(\hat{\mathbf{b}}, \lambda)$ be the boost that boosts a classical particle at rest to a particle with momentum $\mathbf{p}$. Also let $D(\mathbf{p}) = D\big(\Lambda(\mathbf{p})\big)$. This is all exactly as in lecture. Finally, let $u(\mathbf{p}) = D(\mathbf{p})u(\mathbf{0})$ and $v(\mathbf{p}) = D(\mathbf{p})v(\mathbf{0})$. Give a representation independent proof that

$$u(\mathbf{p})^\dagger v(-\mathbf{p}) = 0. \tag{2}$$

This proves Eq. (3.11) in the case that $\epsilon_r \neq \epsilon_{r'}$.

(b) Define $u(\mathbf{p})$ and $v(\mathbf{p})$ as in part (a). Show that

$$u(\mathbf{p})^\dagger u(\mathbf{p}) = v(\mathbf{p})^\dagger v(\mathbf{p}) = \frac{E}{mc^2}. \tag{3}$$

This proves Eq. (3.11) in the case that $r = r'$.

It remains to show that the spinors $w^r(\mathbf{p})$ are Hermitian orthogonal when $\epsilon_r = \epsilon_{r'}$ but $r \neq r'$. This involves the spin which we won't go into since the spin projection operators were not covered in lecture.

3. Consider a Dirac electron ($q = -e$) in a uniform magnetic field, $\mathbf{B} = B_0\hat{\mathbf{z}}$. Choose the gauge,

$$\mathbf{A} = B_0 x\hat{\mathbf{y}}, \tag{4}$$

which is translationally invariant in the y-direction. This means that p_y will be a constant of the motion (although you must distinguish between the kinetic and canonical momentum).

Here is some background on the nonrelativistic problem, which will save you some time. Ignore the spin and the motion in the z-direction, for simplicity. Then the Schrödinger equation is

$$\frac{1}{2m}\Big[\hat{p}_x^2 + \Big(\hat{p}_y + m\omega x\Big)^2\Big]\psi(x,y) = E\psi(x,y). \tag{5}$$

Here we put hats on the momentum operators to distinguish them from the corresponding eigenvalues (where relevant), which are c-numbers. For example, $\hat{p}_y = -i\hbar\partial/\partial y$. Also, we define

$$\omega = \frac{eB_0}{mc}. \tag{6}$$

Then the wave equation (5) is separable, and has the solution,

$$\psi(x,y) = e^{ip_y y/\hbar}\, u_n(\xi), \tag{7}$$

where p_y is the eigenvalue of $\hat{p}_y$, where

$$\xi = x + \frac{p_y}{m\omega}, \tag{8}$$

and where u_n is the usual, normalized Hermite function for the one-dimensional harmonic oscillator with frequency ω,

$$u_n(\xi) = \Big(\frac{m\omega}{\pi\hbar}\Big)^{1/4}\frac{1}{\sqrt{n!2^n}} H_n\Big(\sqrt{\frac{m\omega}{\hbar}}\,\xi\Big)\exp\Big(-\frac{m\omega x^2}{2\hbar}\Big). \tag{9}$$

Here H_n is the usual Hermite polynomial, defined by Eq. (8.44). The energy eigenvalue for the eigenfunction (7) is

$$E = (n + \tfrac{1}{2})\hbar\omega, \tag{10}$$

where $n = 0, 1, 2, \ldots$ is the Landau level. The energy is independent of the quantum number p_y. The wavefunction is like a ridge in the x-y plane, centered on $x = -p_y/m\omega$ (i.e., on $\xi = 0$), and running in the y-direction.

(a) Solve the Dirac equation for the relativistic electron in the same magnetic field. This time you must include the z-motion and the spin. Express the energy in terms of the quantum numbers (n, p_y, p_z, m_s). Write out explicitly a complete set of positive energy solutions as 4-component spinors. You need not normalize these solutions, and you may ignore the negative energy solutions.

(b) Consider the motion of a Dirac electron in the field,

$$\mathbf{B} = B_1 \hat{\mathbf{z}}, \qquad \mathbf{E} = E_1 \hat{\mathbf{x}}, \tag{11}$$

where $0 < E_1 < B_1$. The solution of the Dirac equation for this problem can be obtained from the solution to part (a) by carrying out a Lorentz transformation. Find matrices Λ and $D(\Lambda)$ which will cause $\psi'(x)$ to be the solution in the field (11) if $\psi(x)$ is the solution in the purely magnetic field of part (a). You will also need to find a relation between (E_1, B_1) and B_0. You need not write out the solution $\psi'(x)$ explicitly, but do find the energy eigenvalues in terms of the same quantum numbers (n, p_y, p_z, m_s) as in part (a).