Physics 221B
Spring 2012
Homework 21
Due Friday, March 16, 2012

Reading Assignment:

Rest of Notes 41, and handwritten notes for the week on the Klein-Gordon equation and the Dirac equation; Sakurai, *Advanced Quantum Mechanics*, pp. 23–47.

Sakurai uses the complex coordinate $x_4 = ict$ in his presentation of special relativity. This has the advantage that the Minkowski metric becomes Euclidean, and it is not necessary to distinguish between contravariant and covariant indices. This was the frequently the custom in the older literature on special relativity and the Dirac equation. On the other hand, this approach is unphysical (time is real, not imaginary).

In my lectures I have been following the presentation of Bjorken and Drell, *Relativisitic Quantum Mechanics*, on the Dirac equation. They use real time but the non-Euclidean or Minkowski metric $g_{\mu\nu}$. This is definitely more in fashion nowadays, especially since there is wider interest in general relativity where one must use a non-Euclidean metric. Nevertheless, I will ask you to read over Sakurai's presentation, because his discussion of the physics is good. You should be able to follow in outline what Sakurai does with the Dirac equation, without going into the details of his complex coordinates. With that in mind, please read Sakurai, pp. 75–83.

1. The Dirac equation in two space dimensions. Suppose we lived in a world with two space dimensions (x and y) and one time dimension. Let $x^\mu = (ct, x, y)$, and otherwise use the obvious restrictions of ordinary relativity theory to two spatial dimensions (for example, $g_{\mu\nu} = \text{diag}(+1, -1, -1)$). In two spatial dimensions, the vector potential $\mathbf{A}$ has two components (A_x, A_y), but the magnetic field is a scalar,

$$B = \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}. \tag{1}$$

More generally, true vectors in three dimensions become 2-vectors when restricted to two dimensions, but pseudo-vectors in three dimensions become (pseudo) scalars in two dimensions. You will find that in some respects the Dirac equation in two spatial dimensions is very similar to what was done in lecture in three spatial dimensions, and in other respects it is different.

(a) Carry out Dirac's program of finding a relativistic wave equation that is first order in both space and time. Show that the Dirac algebra of $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$ and β matrices can be satisfied by 2×2 Hermitian matrices. This is the simplest solution for the Dirac algebra in $2 + 1$ dimensional space-time. Thus, the Dirac spinor has two components. The solution is not unique, because you can change the basis (conjugate any solution with some unitary matrix to create another representation of the algebra), so let your primary solution be one in which β is diagonal. Call this the "Dirac-Pauli" representation. Also compute the matrices this the "Dirac-Pauli" representation. Also compute the matrices γ^μ in the Dirac-Pauli representation. The Dirac-Pauli representation is most convenient for studying the nonrelativistic limit.

Now find a representation in which the matrices γ^μ are purely imaginary. Call this the "Maiorana representation." The Maiorana representation makes Lorentz transformations on the spinors look most simple.

In the remaining parts of this problem, carry out your calculations in a representation independent manner, that is, using only the algebraic properties of the matrices in question, unless the problem calls for a specific representation (for example, part (d)).

(b) Write out the Dirac equation including minimal coupling to the electromagnetic field, and show that it possesses a positive definite probability density with an associated probability current that taken together satisfy the continuity equation.

(c) Work out the Heisenberg equations of motion for $\mathbf{x}$ and $\boldsymbol{\pi} = \mathbf{p} - (q/c)\mathbf{A}$.

This problem will be continued next week; please keep a copy of your solution so far so you can refer to it next week.