# Physics 221A
# Fall 2005
# Homework 8
# Due Thursday, October 27, 2005

Reading Assignment: Sakurai pp. 156–174, 187–195, Notes 10, Notes 11.

**1.** The axis $\hat{\mathbf{n}}$ of a rotation $\mathsf{R}$ is a vector that is left invariant by the action of $\mathsf{R}$, that is, it satisfies $\mathsf{R}\hat{\mathbf{n}} = \hat{\mathbf{n}}$. Therefore $\hat{\mathbf{n}}$ is a real, normalized eigenvector of $\mathsf{R}$ with eigenvalue $+1$. Prove that every (proper) rotation has an axis. Show that the axis is unique (apart from the change in sign, $\hat{\mathbf{n}} \to -\hat{\mathbf{n}}$) as long as the angle of rotation is nonzero. This proof is the essential step in showing that every rotation can be expressed in axis-angle form. Do proper rotations in 4 dimensions have an axis?

**2.** Show that if $\mathsf{R} \in SO(3)$, then

$$\mathsf{R}(\mathbf{a}\times\mathbf{b}) = (\mathsf{R}\mathbf{a})\times(\mathsf{R}\mathbf{b}). \tag{1}$$

Hint: Use the fact that if $\mathsf{M}$ is any $3 \times 3$ matrix, then

$$\epsilon_{ijk} \det \mathsf{M} = \epsilon_{\ell mn} M_{i\ell} M_{jm} M_{kn}. \tag{2}$$

This is essentially the definition of the determinant. This proves Eq. (9.36), and hence the adjoint formulas (9.32) and (9.34).

**3.** Show that every $2 \times 2$ matrix of the form,

$$U = e^{-i\theta\hat{\mathbf{n}}\cdot\boldsymbol{\sigma}/2} \tag{3}$$

is an element of the group $SU(2)$. Show that every element of the group $SU(2)$ has the form (3) for some choice of $\theta$, $\hat{\mathbf{n}}$. This problem shows that the spin-$\frac{1}{2}$ rotation operators can be identified with the group $SU(2)$, and that there is no loss of generality in writing a spinor rotation in axis-angle form.

**4.** You are probably aware that the Pauli matrices combined with the $2\times2$ identity matrix span the space of $2 \times 2$ matrices, that is, an arbitrary $2 \times 2$ matrix $A$ can be written in the form,

$$A = a_0 I + \mathbf{a}\cdot\boldsymbol{\sigma}, \tag{4}$$

where $(a_0, \mathbf{a})$ are the (generally complex) expansion coefficients. Notice that if $A$ is Hermitian, then $(a_0, \mathbf{a})$ are real. Notice also that

$$\det A = a_0^2 - \mathbf{a} \cdot \mathbf{a}. \tag{5}$$

It is convenient to write $\sigma_0 = I$, and to regard $\sigma_0$ as a fourth Pauli matrix. Then Eq. (4) becomes,

$$A = \sum_{\mu=0}^{3} a_\mu \, \sigma_\mu. \tag{6}$$

Notice that we have the orthogonality relation,

$$\mathrm{tr}(\sigma_\mu \sigma_\nu) = 2\delta_{\mu\nu}, \tag{7}$$

which can be used to solve Eq. (6) for the expansion coefficients,

$$a_\mu = \frac{1}{2} \, \mathrm{tr}(\sigma_\mu A). \tag{8}$$

Use these results to show that

$$\mathrm{tr}(AB) = \frac{1}{2} \sum_\mu \mathrm{tr}(\sigma_\mu A) \, \mathrm{tr}(\sigma_\mu B), \tag{9}$$

where $A$ and $B$ are $2 \times 2$ matrices.

Use these results to prove Eq. (10.35), that is,

$$\mathsf{R}(U_1)\mathsf{R}(U_2) = \mathsf{R}(U_1 U_2), \tag{10}$$

where $\mathsf{R}(U)$ is defined by Eq. (10.34).

**5.** Sakurai, problem 3.8, p. 243. Also write out the components of $\hat{\mathbf{n}}$. This problem could also be done with $SO(3)$ matrices, but it is a miserable calculation. This is a nice example of how many properties of $SO(3)$ are easier to analyze in terms of the corresponding $SU(2)$ matrices.

**6.** This problem concerns the polarization states of classical electromagnetic waves, which can be described by the same mathematical formalism used for spinors of spin-$\frac{1}{2}$ particles. It is also provides a good background for the subject of the polarization states of photons, which we will take up later in the course.

**(a)** The spinor "pointing in" the $\hat{\mathbf{n}}$ direction was defined by Eq. (10.49). Show that every spinor "points" in some direction. For this, it is sufficient to show that for every normalized spinor $|\chi\rangle$, $\langle\chi|\boldsymbol{\sigma}|\chi\rangle$ is a unit vector. You can prove this directly, or use the formalism based on Eq. (9) above. This property only holds for spin-$\frac{1}{2}$ particles.

**(b)** Consider now the phenomenon of polarization in classical electromagnetic theory. The most general physical electric field of a plane light wave of frequency $\omega$ travelling in the $z$-direction can be written in the form

$$\mathbf{E}_{\rm phys}(\mathbf{r},t) = \mathrm{Re}\sum_{\mu=1}^{2} \hat{\boldsymbol{\epsilon}}_\mu E_\mu e^{i(kz-\omega t)}, \tag{11}$$

where $\mu = 1,2$ corresponds to $x,y$, where $\hat{\boldsymbol{\epsilon}}_1 = \hat{\mathbf{x}}$, $\hat{\boldsymbol{\epsilon}}_2 = \hat{\mathbf{y}}$, and where $E_1 = E_x$ and $E_2 = E_y$ are two complex amplitudes, and where $\omega = ck$. Thus, the wave is parameterized by the two complex numbers, $E_x$, $E_y$. Often we are not interested in absolute amplitudes, only relative ones, so we normalize the wave by setting

$$|E_x|^2 + |E_y|^2 = E_0^2, \tag{12}$$

for some suitably chosen reference amplitude $E_0$. This allows us to associate the wave with a normalized, 2-component complex "spinor," according to

$$\chi = \frac{1}{E_0}\begin{pmatrix} E_x \\ E_y \end{pmatrix}. \tag{13}$$

Furthermore, we are often not interested in any overall phase of this spinor, since such an overall phase corresponds merely to a shift in the origin of time in Eq. (11). This cannot be detected anyway in experiments that average over the rapid oscillations of the wave (a practical necessity at optical frequencies).

If we normalize according to Eqs. (12) and (13) and ignore the overall phase, then the four real parameters originally inherent in $(E_x, E_y)$ are reduced to two, which describe the state of *polarization* of the wave. For example, the spinors

$$\chi_x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \chi_y = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{14}$$

correspond to linear polarization in the $x$- and $y$-directions, respectively. Notice that polarization in the $+x$-direction is the same as that in the $-x$-direction; they differ only by an overall phase. Similarly, the spinors

$$\chi_r = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}, \quad \chi_\ell = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}, \tag{15}$$

correspond to right and left circular polarizations, respectively. Right circular polarization means the electric vector rotates in a clockwise direction in the $x$-$y$ plane, tracing out a circle, whereas in left circular polarization, the electric vector rotates counterclockwise. Sakurai says there is no uniformity in the literature about these conventions, and in fact he uses the opposite conventions for left and right circular polarizations; but the conventions I am quoting here are the ones used by Jackson and Born and Wolf, and I think most physicists use them. Right circular polarization corresponds to photons with negative helicity, and vice versa. In more general polarization states, the electric vector traces out an ellipse in the $x$-$y$ plane; these are called elliptical polarizations. A limiting case of the ellipse is where the electric vector traces a line, back and forth; these are linear polarization states.

In optics it is conventional to introduce the so-called *Stokes' parameters* to describe the state of polarization. These are defined by

$$\begin{aligned} s_0 &= (|E_x|^2 + |E_y|^2)/E_0^2 = 1, \\ s_1 &= (|E_x|^2 - |E_y|^2)/E_0^2, \\ s_2 &= 2\,\mathrm{Re}(E_x E_y^*)/E_0^2, \\ s_3 &= 2\,\mathrm{Im}(E_x E_y^*)/E_0^2. \end{aligned} \tag{16}$$

(See Born and Wolf, *Principles of Optics*, p. 31.) Show that these parameters satisfy

$$s_1^2 + s_2^2 + s_3^2 = 1, \tag{17}$$

so that $\hat{\mathbf{s}} = (s_1, s_2, s_3)$ is a unit vector. The sphere upon which this unit vector lies is called the *Poincaré sphere*; points on this sphere correspond to polarization states. Notice that the Stokes' parameters are independent of the overall phase of the wave, being bilinear in the field amplitudes $(E_x, E_y)$. Indicate which points on the Poincaré sphere correspond to linear $x$- and $y$-polarization, and which to right and left circular polarization. What kind of polarization does the positive 2-axis in $\mathbf{s}$-space correspond to? What about the negative 2-axis? (We will refer to directions in $\mathbf{s}$-space by the indices 1,2,3, to avoid confusion with $x$, $y$, $z$ in real space).

**(c)** Compute the expectation value $\langle\chi|\boldsymbol{\sigma}|\chi\rangle = \hat{\mathbf{n}}$ for the spinor (13), and relate the components of $\hat{\mathbf{n}}$ to the Stokes' parameters. You will see that Stokes and Poincaré didn't exactly follow quantum mechanical conventions (since quantum mechanics had not yet been invented in their day), but the basic idea is that the point on the Poincaré sphere indicates the direction in which the spinor (13) is "pointing."

**(d)** Now suppose we have a quarter-wave plate with its fast and slow axes in the $x$- and $y$-direction, respectively. This causes a relative phase shift in the $x$- and $y$-components of the spinor (13) by $\pi/2$, that is,

$$\begin{pmatrix} E'_x \\ E'_y \end{pmatrix} = \begin{pmatrix} e^{-i\pi/4} & 0 \\ 0 & e^{i\pi/4} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}, \tag{18}$$

where the unprimed fields are those entering the quarter wave plate, and the primed ones are those exiting it. Show that the effect of the quarter wave plate on an incoming polarization state, as represented by a point on the Poincaré sphere, can be represented by a rotation in $\mathbf{s}$-space. Find the $3 \times 3$ rotation matrix such that $\hat{\mathbf{s}}' = \mathsf{R}\hat{\mathbf{s}}$. Use this picture to determine the effect of the quarter wave plate on linear $x$- and $y$- polarizations, and on right and left circular polarizations. What polarization must we feed into the quarter wave plate to get right circular polarization coming out?