Physics 221A
Fall 2005
Homework 7
Due Thursday, October 20, 2005

Reading Assignment: Rest of Notes 8, Notes 9, Sakurai, pp. 123–143 and 152–155.

1. The classical Lagrangian for a particle of charge e in a combined magnetic field and scalar potential V is

$$L(\mathbf{x},\dot{\mathbf{x}}) = \frac{m|\dot{\mathbf{x}}|^2}{2} + \frac{e}{c}\dot{\mathbf{x}}\cdot\mathbf{A}(\mathbf{x}) - V(\mathbf{x}). \tag{1}$$

It turns out that the discretized version of the path integral for the corresponding quantum mechanical particle is

$$K(\mathbf{x},\mathbf{x}_0,t) = \lim_{N\to\infty}\Big(\frac{m}{2\pi i\hbar\epsilon}\Big)^{3N/2}\int d^3\mathbf{x}_1\ldots d^3\mathbf{x}_{N-1}$$
$$\times\exp\Big\{\frac{i\epsilon}{\hbar}\sum_{j=1}^{N}\Big[\frac{m(\mathbf{x}_j-\mathbf{x}_{j-1})^2}{2\epsilon^2}$$
$$+\frac{e}{c}\frac{(\mathbf{x}_j-\mathbf{x}_{j-1})}{\epsilon}\cdot\mathbf{A}\Big(\frac{\mathbf{x}_j+\mathbf{x}_{j-1}}{2}\Big) - V(\mathbf{x}_{j-1})\Big]\Big\}. \tag{2}$$

The interesting thing about this path integral is that the vector potential $\mathbf{A}$ is evaluated at the midpoint of the discretized interval $[\mathbf{x}_{j-1},\mathbf{x}_j]$. Use an analysis like that presented in Sec. 8.11 to show that this discretized path integral is equivalent to the Schrödinger equation for a particle in a magnetic field. Show that this would not be so if the vector potential were evaluated at either end of the interval $[\mathbf{x}_{j-1},\mathbf{x}_j]$ (it must be evaluated at the midpoint). Show that it does not matter which end of the interval the scalar potential is evaluated at.

The delicacy of the points at which the vector potential must be evaluated is related to the fact that the action integrals in the exponent of the Feynman path integral are not really ordinary Riemann sums, because the paths themselves are not differentiable. Instead, they obey the $\Delta x\sim(\Delta t)^{1/2}$ rule discussed in the notes. Casual notation such as Eq. (8.39) glosses over such details.

2. Consider a particle of mass m in a gravitational field, $V(x) = mgx$.

(a) Write out the time-dependent Schrödinger equation for $\psi(x,t)$ in this potential. According to the principle of equivalence, motion in the gravitational fields should be equivalent to free particle motion in a frame which is accelerating downward with acceleration g. Let y be the coordinate in the accelerating frame. Assume the two frames coincide at $t = 0$. Transform the time-dependent Schrödinger equation to the acclerated frame. Now set

$$\psi = \phi e^{i\alpha}, \tag{3}$$

and choose α as a function of space and time such that the Schrödinger equation for ϕ is that of a free particle.

(b) The answer allows any time-dependent free particle solution to be mapped into a time-dependent solution for a particle in a gravitational field. Use this fact and the known propagator for the free particle (see Eq. (8.18)) to write down the propagator for a particle in a gravitational field.

3. This problem is classical mechanics, but I give it anyway because it might help you gain some insight into magnetic monopoles. Consider the motion of an electron of charge $q = -e$ in the field of a magnetic monopole. Assume the monopole is infinitely massive in comparison to the electron. The monopole produces a field,

$$\mathbf{B}(\mathbf{r}) = g\frac{\mathbf{r}}{|\mathbf{r}|^3}, \tag{4}$$

where g is a constant. Although monopoles have never been observed, nevertheless people have carried out experiments to search for them. In such experiments, it is important to know the behavior of ordinary matter in a monopole field, in order to recognize the signatures a monopole would make in experimental apparatus.

(a) Write down Newton's laws for the electron motion. You may use the abbreviation,

$$\mu = \frac{eg}{mc}. \tag{5}$$

To solve these equations of motion, we begin by a search for constants of motion. If it were an electric monopole instead of a magnetic monopole, then we would obviously have four constants of motion: the energy E and angular momentum $\mathbf{L} = m\mathbf{r}\times\mathbf{v}$. Show that the usual (orbital) angular momentum vector $\mathbf{L} = m\mathbf{r}\times\mathbf{v}$ is not conserved. Show that L^2 is conserved, however. This gives you two time-independent constants of motion, (E, L^2), which are not enough to obtain the complete solution. Therefore we must find more constants of motion.

(b) Consider the vector potential,

$$\mathbf{A} = \frac{-g\cos\theta}{r\sin\theta}\hat{\boldsymbol{\phi}}. \tag{6}$$

This vector potential differs by a gauge transformation from the vector potentials dis-ussed in class. It is well behaved everywhere except on the z-axis (both positive and negative). Verify that this vector potential gives the magnetic field of Eq. (4). This vector potential is invariant under rotations about the z-axis, that is, the angle ϕ is ignorable. Use this vector potential in a classical Lagrangian in spherical coordinates, and use Noether's theorem to obtain another conserved quantity (in addition to E, L^2). Show that this quantity is L_z plus another quantity which you may call S_z. Write $J_z = L_z + S_z$, so that J_z is the new conserved quantity. As the notation suggests, interpret J_z as the z-component of a vector $\mathbf{J}$, i.e., guess the formula for $\mathbf{J} = \mathbf{L} + \mathbf{S}$ (all 3 components). Hint: to help you with the guess, transform J_z to rectangular coordinates. By resorting to Newton's laws, show explicitly that $\mathbf{J}$ is conserved. You now have four time-independent constants of motion, $(E, \mathbf{J})$. There are only four, because L^2 is a function of $\mathbf{J}$; show this.

(c) By playing around with dot products of various vectors and looking for exact time derivatives, find more constants of motion. Use these to find r^2 as a function of t. Show that the electron can reach the singularity at $r = 0$ only if $L^2 = 0$.

(d) Show that the orbit lies on a cone whose apex is at the origin. Find the opening angle of the cone as a function of $(E, \mathbf{J})$. Find a vector which specifies the axis of the cone. Assume $L^2 \neq 0$, and let $t = 0$ occur at the point of closest approach. Find the distance of closest approach as a function of (E, L).

(e) Choose the axes so that $\mathbf{J}$ is parallel to the z-axis. Find (r, θ, ϕ) as functions of t.

4. Prove the commutation relations (9.24), using Eq. (9.18) and the properties of the Levi-Civita symbol ϵ_{ijk}.

5. Equation (9.31) was proved in class (and in the notes) by geometrical reasoning. By expressing powers of the J matrices in terms of lower powers, sum the exponential series (9.29) and obtain another proof of Eq. (9.31).