Physics 221A
Fall 2005
Homework 6
Due Thursday, October 13, 2005

Reading Assignment: Notes 8, through Sec. 8.9 (as far as we got in Wednesday's lecture), plus Sakurai, pp. 109–123.

1. A problem on coherent states.

(a) For a one-dimensional problem, work out the commutators, $[W(z_0), \hat{x}]$ and $[W(z_0), \hat{p}]$, where $W(z_0)$ is the Heisenberg operator, defined by Eq. (7.73). Use these to show that the coherent state $|z_0\rangle = W(z_0)|0\rangle$ is an eigenstate of the annihilation operator a with eigenvalue z_0.

(b) The annihilation operator is not Hermitian, so its eigenvalues are not real in general, nor are its eigenstates orthonormal, that is, $\langle z_0|z_1\rangle \neq 0$ when $z_0 \neq z_1$. Nevertheless, they are complete in a certain sense. Prove that

$$\int \frac{dx_0 dp_0}{2\pi} |z_0\rangle\langle z_0| = 1. \tag{1}$$

This resolution of the identity shows that an arbitrary wave function can be expanded as a linear combination of coherent states, that is, Gaussian wave packets. However, the expansion is not unique, because the coherent states are actually overcomplete (they are not linearly independent). In fact, there are many vanishing linear combinations of coherent states (with nonzero coefficients). Nevertheless, the formula (1) looks neat. It can be used as the basis of a path integral.

(c) Use the Heisenberg picture to obtain expressions for $U(t)aU(t)^\dagger$ and $U(t)a^\dagger U(t)^\dagger$, where a and $a^\dagger$ are the usual annihilation and creation operators in the Schrödinger picture, and $U(t)$ is the time-evolution operator for the harmonic oscillator. Then use these to obtain a simple expression for

$$U(t)W(z_0)U(t)^\dagger. \tag{2}$$

Finally, use that to obtain a simple expression for $U(t)|z_0\rangle$. This approach to finding the time evolution of a coherent state is easier than the one taken in the notes, which involved normal ordered Taylor series.

2. The evaluation of the Feynman path integral by the stationary phase approximation requires us to evaluate the determinant of the matrix (8.73), in the limit $N \to \infty$. Fortunately, it can be shown that in this limit, the value of the determinant becomes the solution of a simple differential equation, which has a simple classical interpretation. Furthermore, the result can be expressed in terms of Hamilton's principal function S as shown by Eq. (8.75). This is convenient, because we need S anyway for the phase of the path integral. In this problem we derive Eq. (8.75).

In this problem, as in the lectures and Notes, we denote the final time by t and a variable intermediate time by τ, so that $0 \le \tau \le t$.

(a) A common problem in classical mechanics is to determine the change in the final conditions of some orbit when small changes are made in initial conditions, or vice versa.

Consider a classical particle in 1-dimension, moving in the potential $V(x)$. Suppose some orbit $x(\tau)$ is given; this orbit has certain initial conditions $x_0 = x(0)$, $\dot{x}_0 = \dot{x}(0)$. Suppose we make small changes in the initial conditions, $x_0 \to x_0 + \delta x_0$, $\dot{x}_0 \to \dot{x}_0 + \delta\dot{x}_0$. Denote the resulting orbit by $x(\tau) + \delta x(\tau)$, where $\delta x(\tau)$ is small. Find an approximate equation of motion for $\delta x(\tau)$ by linearizing the equations of motion about the known orbit $x(\tau)$.

Notice that in this problem, $\delta x(\tau)$ represents the difference between two classically allowed orbits, not (as elsewhere in the notes and lectures) a nonphysical variation about a single classical orbit.

(b) Consider the $(N-1) \times (N-1)$ matrix in Eq. (8.73). Call this matrix M. We need $\det M$ in the limit $N \to \infty$. The determinants D_j of the upper diagonal submatrices of M satisfy the recursion relation (8.74); therefore $\det M = D_{N-1}$. Show that for a free particle, $D_j = j + 1$, so $\det M = N$. This suggests (correctly) that for any kinetic-plus-potential problem, $\det M$ will diverge as N as $N \to \infty$, so let us scale variables and write

$$F_j = \epsilon D_j = \frac{t}{N} D_j. \tag{3}$$

In this way, F_{N-1} will approach a well defined limit as $N \to \infty$. Now write $\tau_j = j\epsilon$, $F_j = F(\tau_j)$, and take the limit $N \to \infty$. Show that $F(\tau)$ satisfies the same differential equation as $\delta x(\tau)$ in part (a) of this problem. Also show that $F(\tau)$ satisfies the initial conditions, $F(0) = 0$, $F'(0) = 1$.

The idea contained by this part of the calculation (correctly) is that the determinant (7.75) contains information about the fluctuations about the classical orbit.

(c) Now return to part (a), and assume $\delta x(0) = 0$, $\delta\dot{x}(0) = \delta\dot{x}_0$. Show that

$$\delta x(t) = -m\Big(\frac{\partial^2 S}{\partial x \partial x_0}\Big)^{-1}\delta\dot{x}_0, \tag{4}$$

where $S = S(x, x_0, t)$ and where t is the final time. Hint: Since $\delta x(t)$ is the small variation in final condition due to small variation $\delta\dot{x}_0$ in initial velocity, we have

$$\delta x(t) = \Big(\frac{\partial x}{\partial \dot{x}_0}\Big)_{x_0}\delta\dot{x}_0, \tag{5}$$

where the final position $x = x(t)$ is considered a function of $(x_0, \dot{x}_0)$ as well as t. Then use the generating function relations (8.45).

Finally, explain how (4) can be used to prove

$$\lim_{N\to\infty} \epsilon \det \frac{\partial^2\varphi}{\partial x_k \partial x_\ell} = -m\Big(\frac{\partial^2 S}{\partial x \partial x_0}\Big)^{-1}, \tag{6}$$

which is essentially Eq. (8.75).

3. (a) Sakurai (p. 116) says that there is a unique classical path connecting two endpoints and endtimes, (x_0, t_0), (x, t). For the classical harmonic oscillator with Lagrangian,

$$L = \frac{m\dot{x}^2}{2} - \frac{m\omega^2 x^2}{2}, \tag{7}$$

find values of (x, x_0, t) such that there exists a unique path; no path at all; more than one path. Take $t_0 = 0$, $t_1 = t$ and use τ for a variable intermediate time, $0 \le \tau \le t$, as in the previous problem.

(b) Compute Hamilton's principal function $S(x, x_0, t)$ for the harmonic oscillator, and verify the generating function relations,

$$p_1 = \frac{\partial S}{\partial x_1}, \qquad p_0 = -\frac{\partial S}{\partial x_0}, \qquad H = -\frac{\partial S}{\partial t}. \tag{8}$$

Do this for some time t such that there exists only one classical path.

(c) Sakurai (p. 117) says that the classical path is that which minimizes the action. Let $x(\tau)$ be a classical orbit in the harmonic oscillator, satisfying $x(0) = x_0$, $x(t) = x$ for given values of (x, x_0, t). Consider a modified path, $x(\tau) + \delta x(\tau)$, where of course $\delta x(\tau)$ vanishes at $\tau = 0$ and $\tau = t$. Expand the action along the modified path out to second order in δx, and show that the second variation in the action can be written,

$$\int_0^t d\tau\, \delta x(\tau) K \delta x(\tau), \tag{9}$$

where K is some operator. Find an expression for the operator K, find its eigenvalues κ_n and eigenfunctions $\xi_n(\tau)$. Show that if $t < \pi/\omega$, then all eigenvalues are positive, and the action is a minimum. For other values of time t, show that the number of negative eigenvalues of K is the largest integer less than $\omega t/\pi$. Thus, for $t > \pi/\omega$, the classical path is not a minimum of the action functional, but rather a saddle point.

(d) Put the pieces together, and write out the Van Vleck expression for the propagator of the harmonic oscillator, $K(x, x_0, t)$.

(e) Think of the complex time plane, and consider $K(x, x_0, t)$ for times on the real axis satisfying $0 < t < \pi/\omega$. Analytically continue the expression for K in this time interval down onto the negative imaginary time axis, set $t = -i\hbar\beta$, and get an expression for the density matrix $\langle x|\rho|x_0\rangle$ for a harmonic oscillator in thermal equilibrium. Take the trace of ρ to get the partition function $Z(\beta)$.