Physics 221A
Fall 2005
Homework 3
Due Thursday, September 22, 2005

Reading Assignment: Sakurai, pp. 41–60, 174–187, Notes 3, Notes 4.

1. The projection postulate of quantum mechanics says that if a system is described by a pure state $|\psi\rangle$ (here assumed to be normalized), then after a measurement of the operator A producing eigenvalue a_n, the system is described by the (normalized) state

$$|\psi'\rangle = \frac{P_n|\psi\rangle}{\sqrt{\langle\psi|P_n|\psi\rangle}}, \tag{1}$$

where P_n is the projector onto the n-th eigenspace of A.

Suppose instead the system is described initially by a density operator ρ (assumed normalized). What is the (normalized) density operator ρ' after the measurement? Do not assume the eigenvalue a_n is nondegenerate.

2. Show that if ρ is a density operator, then $\operatorname{tr}\rho^2 \leq 1$, with equality if and only if ρ represents a pure state.

3. Here is another approach to the density operator. Suppose we have a total system t which is composed of a subsystem s of interest, plus the rest r of the total system. For example, let system s be an atom. Then system r could be another microscopic system which has interacted with the atom; it could be a second, identical atom, with both atoms flying apart and carrying spin correlations as in an Einstein-Podolsky-Rosen experiment; it could be a heat bath; it could be a macroscopic measuring apparatus; it could be the rest of the universe. System r may interact with system s only for certain time intervals, as we wish (in other words, we may wish to assume that systems s and r are noninteracting at certain times).

Let $|n\rangle^{(s)}$ and $|m\rangle^{(r)}$ be bases of kets in the Hilbert spaces for the subsystem of interest and the rest of the system, respectively, so that the product kets $|n\rangle^{(s)}|m\rangle^{(r)}$ are basis kets for the Hilbert space for the total system. Let spaces $\mathcal{E}_s$, $\mathcal{E}_r$ and $\mathcal{E}_t$ be the Hilbert spaces for the system of interest, the rest of the system, and the total system, respectively.

Suppose the total system is in a pure state $|\psi\rangle \in \mathcal{E}_t$. If systems s and r are noninteracting (at some time), you might suppose that we could describe system s by means of a wave function, i.e., as a pure state; but this is impossible in general. Show that if A_s is an operator which acts only on ket space $\mathcal{E}_s$, then the expectation value of A_s can be expressed in terms of an operator ρ_s which also acts only on $\mathcal{E}_s$. That is, show that with a proper definition of ρ_s, we have

$$\langle A_s \rangle = \mathrm{tr}(\rho_s A_s). \tag{2}$$

Express ρ_s in terms of the expansion coefficients of $|\psi\rangle$ with respect to a basis in $\mathcal{E}_t$. Show that ρ_s satisfies all the requirements of a density operator, namely, it is Hermitian, nonnegative definite, and has unit trace.

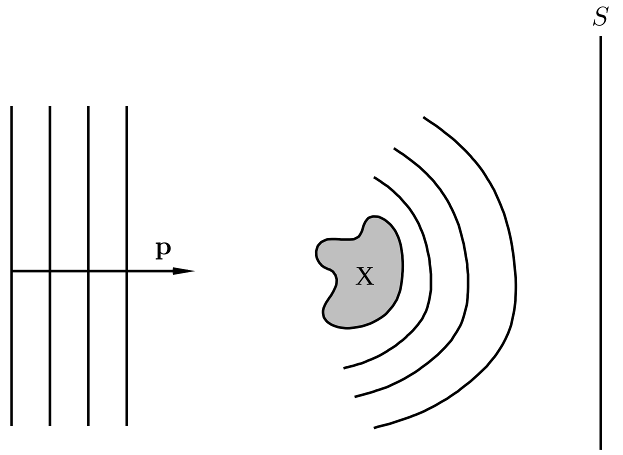

Figure for problem 4. A scattering experiment.

4. How do we measure the wave function $\psi(\mathbf{x})$ of a particle, including its phase? (The particle is assumed to be spinless.) Consider a gedankenexperiment in which electrons of a fixed energy are scattered off some scatterer X. (See figure.) A screen S is placed downstream from the scatterer, and $|\psi(\mathbf{r})|^2$ is measured on the screen by counting electron impacts. The electrons are described by an incident plane wave of the given energy; the beam is of low density, so the electrons do not interact with each other.

Describe a modification to this gedankenexperiment by which the phase of the wave function $\psi(\mathbf{r})$ can be measured on the screen, apart from the overall phase, which of course is nonphysical and can never be measured.

5. Let $|\psi\rangle$ be the state of a spinless particle in three dimensions, and let $\phi(\mathbf{p}) = \langle \mathbf{p}|\psi\rangle$ be its momentum space wave function. Find the momentum space wave function of the state $T(\mathbf{a})|\psi\rangle$, that is, find the action of the translation operator $T(\mathbf{a})$ in the momentum representation.

6. In this problem we denote operators with a hat, as in $\hat{x}$ or $\hat{A}$, and we denote eigenvalues or classical quantities without a hat, as in x or $A(x,p)$. We work in one dimension, and think of a wave function $\psi(x)$ or $\psi(x,t)$.

If $\hat{A}$ is an operator, we define the *Weyl transform* of $\hat{A}$, denoted $A(x,p)$, by

$$A(x,p) = \int_{-\infty}^{+\infty} ds\, e^{-ips/\hbar}\, \langle x+s/2|\hat{A}|x-s/2\rangle. \tag{4}$$

Here the notation $|x-s/2\rangle$, for example, means the eigenket of $\hat{x}$ with eigenvalue $x-s/2$. It is useful to think of $A(x,p)$ as a function defined on the classical (x,p) phase space which is in some sense the classical observable corresponding to the quantum operator $\hat{A}$.

(a) Show that if $A(x,p)$ is the Weyl transform of operator $\hat{A}$, then $A(x,p)^*$ is the Weyl transform of $\hat{A}^\dagger$. In particular, this shows that the Weyl transform of a Hermitian operator is a real function on phase space.

(b) Show that if operators $\hat{A}$ and $\hat{B}$ have Weyl transforms $A(x,p)$ and $B(x,p)$, respectively, then

$$\mathrm{tr}(\hat{A}^\dagger\hat{B}) = \int \frac{dx\,dp}{2\pi\hbar}\, A(x,p)^* B(x,p). \tag{5}$$

Notice how the right hand side looks like the "scalar product" of two classical observables on phase space.

(c) Find the Weyl transforms of the following operators: 1 (the identity operator); $\hat{x}$; $\hat{p}$; $\hat{x}\hat{p}$; $\hat{p}\hat{x}$; $\hat{p}^2/2m + V(\hat{x})$.