Physics 221A
Fall 2005
Homework 12
Due Monday, November 28, 2005

Note: This homework is due the Monday after Thanksgiving, but there will be another due as usual on the following Thursday.

Reading Assignment: Sakurai pp. 271–282, Notes 16.

1. It is shown in Notes 16 that the time reversal operator satisfies

$$\Theta|sm\rangle = \eta i^{2m}|s,-m\rangle. \tag{1}$$

Find a formula for $\Theta^\dagger|sm\rangle$. When is $\Theta = \Theta^\dagger$?

2. This is essentially Sakurai's problem 4.4, p. 282, but I've rewritten it to make it more clear.

We did not discuss the spin angular functions in class, but these are spinor functions on the unit sphere that arise when we combine orbital and spin angular momentum for a central force problem for a spinning particle. Here we will take the case $s = \frac{1}{2}$ (for example, in hydrogen). Let $|n\ell m_\ell\rangle$ be ket language for the wave function $R_{n\ell}(r)Y_{\ell m_\ell}(\theta,\phi)$, the solution of the Schrödinger equation for a spinless particle in a central force field, and let $|sm_s\rangle$ be the usual spin states (here $s = \frac{1}{2}$ and $m_s = \pm\frac{1}{2}$). We distinguish between m_ℓ and m_s, the two types of magnetic quantum numbers. We multiply the wave functions times the spin functions and form linear combinations with the Clebsch-Gordan coefficients to get eigenstates of J^2 and J_z, where $\mathbf{J} = \mathbf{L} + \mathbf{S}$. These are

$$|n\ell j m_j\rangle = \sum_{m_\ell, m_s} |n\ell m_\ell\rangle |sm_s\rangle \langle \ell s m_\ell m_s | j m_j\rangle. \tag{2}$$

The spatial wave functions $\psi_{n\ell m_\ell}(\mathbf{r}) = R_{n\ell}(r)Y_{\ell m_\ell}(\theta,\phi)$ factor into a radial part times an angular part. Let us write this in ket language as $|n\ell m_\ell\rangle = |n\ell\rangle|\ell m_\ell\rangle$. The factor $R_{n\ell}(r)$ or $|n\ell\rangle$ is the same for all terms in the sum above, so it can be taken out and what is left is a two-component spinor that depends only on the angles. This is what in wave function language Sakurai calls $\mathcal{Y}_\ell^{jm_j}$; in ket language we will write

$$|\ell j m_j\rangle = \sum_{m_\ell m_s} |\ell m_\ell\rangle |sm_s\rangle \langle \ell s m_\ell m_s | j m_j\rangle. \tag{3}$$

(a) For the case $\ell = 0$, $j = \frac{1}{2}$, $m_j = \frac{1}{2}$, write out the two-component spinor $\mathcal{Y}_\ell^{jm_j}$ as functions of (θ, ϕ).

(b) Multiply this by $\sigma \cdot \mathbf{r}$ ($\mathbf{r} =$ Sakurai's $\mathbf{x}$), and express the result as a linear combination of other spin angular functions $\mathcal{Y}_\ell^{jm_j}$.

(c) Certain values of j, m_j and ℓ occur in the sum, and certain others do not. Use symmetry principles to explain why the values that occur are allowed and the others are not.
Hint: It may help to think of three kinds of rotations: spin rotations, orbital rotations, and total (spin plus orbital) rotations.

3. Sakurai, problem 4.5, p. 282. When he says further restrictions on the quantum numbers, he means restrictions that go beyond those due to symmetry considerations.

4. This is a variation on Sakurai's problem 4.10, which is confused because you can't prove what he asks you to prove in part (c) (it involves a phase convention). Do this instead:

(a) Let $U(\mathsf{R})$ be a rotation operator on the state space of any system (however complex). Show that $[\Theta, U(\mathsf{R})] = 0$ for all R.

(b) Suppose we are only interested in the spin degrees of freedom of a single particle. Denote the basis states by $|jm\rangle$. By considering $\Theta U(\mathsf{R})|jm\rangle$, show that

$$[D^j_{m'm}(\mathsf{R})]^* = (-1)^{m-m'} D^j_{-m',-m}(\mathsf{R}). \tag{4}$$

(c) Show that if T^k_q is an irreducible tensor operator, then so is

$$S^k_q = (-1)^q \left(T^k_{-q}\right)^\dagger. \tag{5}$$

Operator S^k is regarded as the Hermitian conjugate of T^k.

5. Consider the helium atom in a lab frame. The positions of the nucleus, electron 1 and electron 2 are $\mathbf{r}_n$, $\mathbf{r}_{e1}$ and $\mathbf{r}_{e2}$ respectively. The mass of the nucleus is M and that of the electron is m. Thus the laboratory Hamiltonian is

$$H = \frac{|\mathbf{p}_n|^2}{2M} + \frac{|\mathbf{p}_{e1}|^2}{2m} + \frac{|\mathbf{p}_{e2}|^2}{2m} + V(\mathbf{r}_n, \mathbf{r}_{e1}, \mathbf{r}_{e2}). \tag{6}$$

We will only be interested in the kinetic energy in this problem.

Let $\mathbf{R}$ be the center of mass position and let

$$\begin{aligned} \mathbf{r}_1 &= \mathbf{r}_{e1} - \mathbf{r}_n, \\ \mathbf{r}_2 &= \mathbf{r}_{e2} - \mathbf{r}_n, \end{aligned} \tag{7}$$

so that $\mathbf{r}_1$ and $\mathbf{r}_2$ are the positions of the two electrons relative to the nucleus. Also define new momentum operators,

$$\mathbf{P} = -i\hbar\frac{\partial}{\partial\mathbf{R}}, \quad \mathbf{p}_1 = -i\hbar\frac{\partial}{\partial\mathbf{r}_1}, \quad \mathbf{p}_2 = -i\hbar\frac{\partial}{\partial\mathbf{r}_2}. \tag{8}$$

Express the kinetic energy as a function of these new momentum operators. You will obtain a term proportional to $\mathbf{p}_1 \cdot \mathbf{p}_2$. This is called a "mass polarization" term.

The usual simple minded treatment of helium treats the nucleus as infinitely heavy. In this approach there are no mass polarization terms. Explain why its a good approximation to neglect those terms (thus, the usual approach is ok).