Physics 221A
Fall 2005
Homework 10
Due Thursday, November 10, 2005

Reading Assignment: Sakurai pp. 203–217, 232–233, Notes 13, 14.

1. A problem concerning electrons in magnetic fields.

(a) In problem 2 of homework 5, you worked out the energy levels and energy eigenfunctions for an electron in a uniform magnetic field, but we ignored the spin. I said this was not a good approximation, because if you include the spin it does more than just make small perturbations in the levels you found without spin.

Including the spin, find the energy levels of an electron in a uniform magnetic field. Express your answer in terms of $\omega_0 = eB/mc$, the orbital frequency of a classical electron in a uniform magnetic field.

Hint: This is a short problem, and does not require any lengthy calculations (unlike that earlier problem).

(b) Consider an electron in a central force potential $V(r)$, plus a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{b}}$. Include the interaction of the spin with the magnetic field (but ignore fine structure). Again let $\omega_0 = eB/mc$. Use the gauge $\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r}$. The Schrödinger equation is

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = H|\psi(t)\rangle. \tag{1}$$

Define a new state $|\phi(t)\rangle$ by

$$|\psi(t)\rangle = U(\hat{\mathbf{b}}, \omega t)|\phi(t)\rangle, \tag{2}$$

where $U(\hat{\mathbf{b}}, \omega t)$ is a rotation operator. This means that $|\phi(t)\rangle$ is the state in a frame rotating with angular velocity ω about the axis $\hat{\mathbf{b}}$.

Find a frequency ω that eliminates the effect of the magnetic field on the orbital motion of the particle, apart from an extra potential proportional to $(\mathbf{b}\times\mathbf{r})^2$. Find a frequency ω that eliminates the effect of the magnetic field on the spin. Espress your answers as some multiple of ω_0.

2. Consider a particle of spin 1 and magnetic moment $\boldsymbol{\mu} = -\gamma\mathbf{S}$ in the magnetic field,

$$\mathbf{B} = B_0\hat{\mathbf{z}} + B_{10}(\hat{\mathbf{x}}\cos\omega_1 t + \hat{\mathbf{y}}\sin\omega_1 t), \tag{3}$$

employed in magnetic resonance experiments (assume $\gamma > 0$). If at $t = 0$, the particle is in state $m = 0$, find the transition probabilities $P(0 \to \pm 1)$ as a function of time.

3. If we combine Eq. (13.22) with (13.29), we obtain

$$\frac{\partial U}{\partial t} = -\frac{i}{\hbar}\boldsymbol{\omega}(t) \cdot \mathbf{S}\, U, \tag{4}$$

where we write U instead of T for the time evolution operator, which we know is a rotation. Let U be parameterized by its Euler angles, $U = U(\alpha, \beta, \gamma)$. Find equations of motion for the Euler angles, assuming $\boldsymbol{\omega}(t)$ is given. Your answer will be identical to the equations of motion of the Euler angles in classical rigid body theory (for given $\boldsymbol{\omega}(t)$).

4. Consider a biological sample at 300K in a magnetic field of 6T (for example, you in an MRI device). After a certain relaxation time, the nuclear spins will reach thermal equilibrium with their environment (a heat bath). Calculate the fractional magnetization of protons under such circumstances (the magnetization compared to the maximum we would have at 0K). Finally, for a sample of water under the conditions indicated, compute the magnetization due to protons in Gauss.

5. I won't ask you to work out any numerical values of Clebsch-Gordan coefficients, but you should be comfortable in doing so. The following is a related problem.

Consider the angular momentum problem $\ell \otimes \frac{1}{2}$, where ℓ is arbitrary and could be very large. We write $j = \ell + \frac{1}{2}$ or $j = \ell - \frac{1}{2}$ for the resulting angular momentum. By beginning with the doubly stretched state,

$$|\ell + \tfrac{1}{2}, \ell + \tfrac{1}{2}\rangle = |\ell\ell\rangle|\tfrac{1}{2}\tfrac{1}{2}\rangle, \tag{5}$$

apply lowering operators to construct the states $|\ell + \frac{1}{2}, m\rangle$ for m going down to $\ell - \frac{5}{2}$. By this time a pattern should be evident; guess it, and prove that it is right by induction, to obtain a general formula for $|\ell + \frac{1}{2}, m\rangle$. You may simplify notation by omitting the total angular momenta ℓ and $\frac{1}{2}$ in the kets on the right hand sides of your equations, because they are always the same; for example, the RHS of Eq. (5) can be written simply $|\ell\rangle|\frac{1}{2}\rangle$.

Now construct the stretched state for $j = \ell - \frac{1}{2}$, namely $|\ell - \frac{1}{2}, \ell - \frac{1}{2}\rangle$. Use the standard phase convention given by Eq. (14.31). Then lower this enough times to see a pattern, guess it, and prove it by induction. Note the Useful Formula section below.

Useful Formulas

The following are some useful formulas, which can be derived by the methods of problem 5. We combine $\ell \otimes 1$, and find, for the three cases $j = \ell + 1$, $j = \ell$, and $j = \ell - 1$, the following:

$$|\ell + 1, m\rangle = \sqrt{\frac{(\ell + m + 1)(\ell + m)}{(2\ell + 2)(2\ell + 1)}}\, |m - 1\rangle|1\rangle$$

$$+ \sqrt{\frac{(\ell - m + 1)(\ell + m + 1)}{(\ell + 1)(2\ell + 1)}}\, |m\rangle|0\rangle + \sqrt{\frac{(\ell - m)(\ell - m + 1)}{(2\ell + 2)(2\ell + 1)}}\, |m + 1\rangle| - 1\rangle. \tag{6a}$$

$$|\ell m\rangle = -\sqrt{\frac{(\ell - m + 1)(\ell + m)}{2\ell(\ell + 1)}}\, |m - 1\rangle|1\rangle$$

$$+ \frac{m}{\sqrt{\ell(\ell + 1)}}|m\rangle|0\rangle + \sqrt{\frac{(\ell - m)(\ell + m + 1)}{2\ell(\ell + 1)}}\, |m + 1\rangle| - 1\rangle. \tag{6b}$$

$$|\ell - 1, m\rangle = \sqrt{\frac{(\ell - m)(\ell - m + 1)}{2\ell(2\ell + 1)}}\, |m - 1\rangle|1\rangle$$

$$- \sqrt{\frac{(\ell - m)(\ell + m)}{\ell(2\ell + 1)}}\, |m\rangle|0\rangle + \sqrt{\frac{(\ell + m + 1)(\ell + m)}{2\ell(2\ell + 1)}}\, |m + 1\rangle| - 1\rangle. \tag{6c}$$