Physics 221A
Fall 2005
Homework 1
Due Saturday, September 10, 2005

Reading Assignment: Sakurai, pp. 1–23, Notes 1.

1. Consider the 2×2 matrices:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1}$$

(a) Prove that

$$\exp(i\theta\boldsymbol{\sigma}\cdot\hat{\mathbf{n}}) = I\cos\theta + i(\boldsymbol{\sigma}\cdot\hat{\mathbf{n}})\sin\theta, \tag{2}$$

where

$$\boldsymbol{\sigma} = \sigma_x\hat{\mathbf{x}} + \sigma_y\hat{\mathbf{y}} + \sigma_z\hat{\mathbf{z}}. \tag{3}$$

Here $\hat{\mathbf{n}}$ is an arbitrary unit vector, and θ an arbitrary angle.

(b) Prove that, given any two vector operators $\mathbf{A}$, $\mathbf{B}$ that commute with $\boldsymbol{\sigma}$ (but not necessarily with each other), we have the identity,

$$(\boldsymbol{\sigma}\cdot\mathbf{A})(\boldsymbol{\sigma}\cdot\mathbf{B}) = \mathbf{A}\cdot\mathbf{B} + i\boldsymbol{\sigma}\cdot(\mathbf{A}\times\mathbf{B}). \tag{4}$$

Note that in general, $\mathbf{A}\cdot\mathbf{B} \neq \mathbf{B}\cdot\mathbf{A}$, and $\mathbf{A}\times\mathbf{B} \neq -\mathbf{B}\times\mathbf{A}$.

2.(a) Consider two operators A, B that do not necessarily commute. Show that

$$e^A B e^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + \frac{1}{3!}[A,[A,[A,B]]] + \ldots. \tag{5}$$

Hint: Replace A by λA, where λ is a parameter, and let the left-hand side be $F(\lambda)$. Find a differential equation satisfied by $F(\lambda)$, and solve it.

(b) Let $A(\lambda)$ be an operator that depends on a continuous parameter λ. Derive the following operator identity:

$$e^{-iA}\frac{d(e^{iA})}{d\lambda} = i\sum_{n=0}^{\infty}\frac{(-i)^n}{(n+1)!}L_A^n\left(\frac{dA}{d\lambda}\right), \tag{6}$$

where

$$L_A(X) = [A, X], \quad L_A^2(X) = [A, [A, X]], \quad \ldots, \tag{7}$$

where X is an arbitrary operator.

In the following problems, you may assume wherever necessary that you are dealing with a finite-dimensional Hilbert space.

3. Show that if a linear operator A satisfies any two of the following three conditions, it satisfies the third: (a) A is Hermitian; (b) A is unitary; (c) $A^2 = 1$.

4. Some easy proofs from Notes 1.

(a) Show that Eq. (1.42) follows from Eq. (1.41).

(b) Prove Eqs. (1.49), (1.52), and (1.53).

(c) Prove that the product of two Hermitian operators is Hermitian if and only if they commute.

5. In general, there is little interest in the eigenkets and eigenbras of an arbitrary operator. Hermitian operators are an exception; so are anti-Hermitian and unitary operators. We define an operator to be *normal* if it commutes with its Hermitian conjugate, $[A, A^\dagger] = 0$. Notice that Hermitian, anti-Hermitian, and unitary operators are normal.

(a) Show that if A is normal, and $A|u\rangle = a|u\rangle$ for some nonzero $|u\rangle$, then $A^\dagger|u\rangle = a^*|u\rangle$. Thus, the eigenbras of A are the Hermitian conjugates of the eigenkets, and the left spectrum is identical to the right spectrum. Hint: it is not necessary to introduce orthonormal bases or anything of the kind.

(b) Show that the eigenspaces corresponding to distinct eigenvalues of a normal operator are orthogonal. This is a generalization of the easy and familiar proof for Hermitian operators.