Physics 209
Fall 2002
Notes 5
Thomas Precession

Jackson's discussion of Thomas precession is based on Thomas's original treatment, and on the later paper by Bargmann, Michel, and Telegdi. The alternative treatment presented in these notes is more geometrical in spirit and makes greater effort to identify the aspects of the problem that are dependent on the state of the observer and those that are Lorentz and gauge invariant. There is one part of the problem that involves some algebra (the calculation of Thomas's angular velocity), and this is precisely the part that is dependent on the state of the observer (the calculation is specific to a particular Lorentz frame). The rest of the theory is actually quite simple. In the following we will choose units so that $c = 1$, except that the c's will be restored in some final formulas.

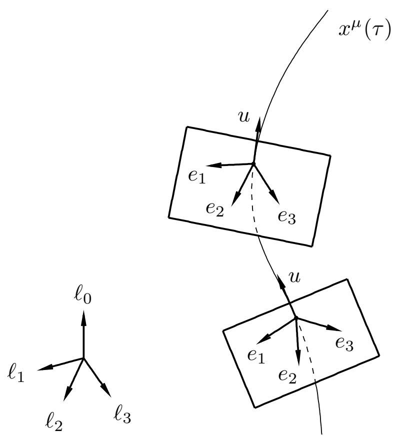

Fig. 5.1. Lab frame $\{\ell_\alpha\}$ and conventional rest frames $\{e_\alpha\}$ along the world line $x^\mu(\tau)$ of a particle. The time-like basis vector e_0 of the conventional rest frame is the same as the world velocity u of the particle. The spatial axes of the conventional rest frame $\{e_i, i = 1, 2, 3\}$ span the three-dimensional space-like hyperplane orthogonal to the world line.

To begin we must be careful of the phrase "the rest frame of the particle," which is used frequently in relativity theory and in the theory of Thomas precession. The geometrical situation is indicated schematically in Fig. 5.1, which shows the world line $x^\mu = x^\mu(\tau)$ of a particle (τ is the proper time). The world line is curved, indicating that the particle is accelerated. The world velocity of the particle is $u^\mu = dx^\mu/d\tau$. We will frequently write this vector as simply u (without the μ superscript), and similarly for other vectors, when there is no danger of confusion. The world velocity is a time-like unit vector,

$$u^\mu u_\mu = (u \cdot u) = 1, \tag{5.1}$$

where we introduce an obvious notation for the Minkowski scalar product of two vectors, and it is tangent to the world line of the particle. We recall that the world velocity is also Minkowski orthogonal to the world acceleration $b^\mu = du^\mu/d\tau$,

$$u^\mu b_\mu = (u \cdot b) = 0, \tag{5.2}$$

which follows from differentiating Eq. (5.1) with respect to τ. Here we use the symbol b for the world acceleration, since the symbol a or $\mathbf{a} = d\mathbf{v}/dt$ will be reserved for the ordinary 3-acceleration.

The figure also shows the tetrad of unit vectors $\{\ell_\alpha, \alpha = 0, 1, 2, 3\}$ that define a "lab" frame of some observer. Notice that the subscripts on these vectors are not component indices, but serve only to label the vectors. The vector ℓ_0 is a unit time-like vector defining the time-axis of the observer, and the vectors ℓ_i, $i = 1, 2, 3$ are the unit space-like vectors defining the observer's x, y, z axes. These unit vectors are orthonormal in the sense

$$(\ell_\alpha \cdot \ell_\beta) = \ell^\mu_\alpha \, g_{\mu\nu} \, \ell^\nu_\beta = g_{\alpha\beta}, \tag{5.3}$$

where, for example, ℓ^μ_α means the μ-th contravariant component of the vector ℓ_α. These components can be measured in any Lorentz frame, but if we use the Lorentz frame defined by the vectors $\{\ell_\alpha\}$ themselves, then $\ell^\mu_\alpha = \delta^\mu_\alpha$.

Let X be any vector, and X^μ its components in the lab frame. The relation between the vector and its components is

$$X = X^\alpha \ell_\alpha. \tag{5.4}$$

In this equation notice that X and ℓ_α are vectors, while the X^α are numbers. To find the components in terms of the vector, we take the dot product of Eq. (5.4) with ℓ_β and use Eq. (5.3) to obtain

$$(\ell_\beta \cdot X) = g_{\alpha\beta} \, X^\alpha = X_\beta. \tag{5.5}$$

Equations (5.4) and (5.5) allow us to go back and forth between a vector and its components.

When we speak of a rest frame of a particle, we are referring to a frame in which the particle is instantaneously at rest. For an accelerated particle, this frame must be a function of τ. For an unaccelerated particle, the world line of the particle coincides with the time axis of the rest frame, so for an accelerated particle, we require that the time axis of the instantaneous rest frame be tangent to the world line. That is, we require that the unit vector defining the time axis of the instanteous rest frame be the world velocity u. As for the three spatial axes of the instantaneous rest frame, there is nothing in general to dictate any privileged choice for these, apart from the obvious fact that they should be orthogonal to u. (We may also wish to require that they be right-handed.) Thus, the spatial unit vectors can be chosen to be any three orthonormal, space-like unit vectors that span the three-dimensional hyperplane perpendicular to u (in the Minkowski sense). In Fig. 5.1 the basis vectors of the rest frame of the particle are denoted $\{e_\alpha, \alpha = 0, 1, 2, 3\}$, with $e_0 = u$. The spatial unit vectors $\{e_i, i = 1, 2, 3\}$ are chosen according to some arbitrary convention. Notice that there is a different choice of spatial unit vectors to be made at each point of the world line of the particle, since the vector u is changing. Because of the arbitrariness in the choice of the spatial axes, we will call the rest frame $\{e_\alpha\}$ a *conventional rest frame.*

Any two choices of a conventional frame at a given point on the world line of the particle differ from one another by a purely spatial rotation, since the time axis is determined. That is, if $\{e_i\}$ and $\{e'_i\}$ are two choices of spatial axes of a rest frame, they must be related by

$$e'_i = R_{ij}\, e_j, \tag{5.6}$$

where R_{ij} is a rotation matrix (a proper rotation, if we wish to preserve the handedness of the spatial axes). Notice that R_{ij} is in general a function of τ. Obviously there are an infinite number of ways of choosing a conventional rest frame along the world line of a particle, and this is why we must be careful about the expression, "the" rest frame of the particle.

In traditional approaches to Thomas precession, there is one particular rest frame that is usually referred to. This is the rest frame that is obtained from the lab frame $\{\ell_\alpha\}$ by doing a pure boost in the direction of the velocity of the particle, as seen in the lab frame. This is an example of a conventional rest frame, and its arbitrary nature can be seen from the fact that another observer, moving with respect to the first observer (whose stationary frame is the lab frame $\{\ell_\alpha\}$), would obtain a different conventional rest frame by doing a pure boost from *his* rest or "lab" frame. Obviously the particle that is following its world line does not know which observer is observing it, so these different choices of conventional rest frame have nothing to do with any intrinsic property of the dynamics.

Let us call the conventional rest frame obtained from the lab frame by a pure boost the "Thomas conventional frame." If X is an arbitrary vector with components X^μ and X'^μ in the lab and Thomas conventional frames, respectively, then these are related by the

Lorentz transformation,

$$X'^{\mu} = L^{\mu}{}_{\nu}(\mathbf{v})X^{\nu}, \tag{5.7}$$

where $L^{\mu}{}_{\nu}(\mathbf{v})$ is the pure boost specified by the velocity $\mathbf{v}$, the velocity of the particle as seen in the lab frame at some instant along its world line. Actually, in the following we will need the components of the inverse transformation, which is obtained by changing $\mathbf{v}$ to $-\mathbf{v}$ (since $L(\mathbf{v})^{-1} = L(-\mathbf{v})$), so that

$$X^{\mu} = L^{\mu}{}_{\nu}(-\mathbf{v})X'^{\nu}. \tag{5.8}$$

These components are obtained explicitly if we write out the Lorentz transformation between the two frames,

$$t = \gamma(t' + \mathbf{v}\cdot\mathbf{x}'), \tag{5.9a}$$

$$\mathbf{x} = \mathbf{x}' + (\gamma - 1)\frac{(\mathbf{v}\cdot\mathbf{x}')\mathbf{v}}{v^2} + \gamma\mathbf{v}t', \tag{5.9b}$$

where the unprimed coordinates are measured in the lab frame and the primed coordinates in the Thomas conventional frame. If Eqs. (5.9) are written out in matrix-vector form, the components of $L^{\mu}{}_{\nu}(-\mathbf{v})$ can be read off.

Now let the basis unit vectors of the Thomas conventional frame be e_α, with components e'^{μ}_{α} and e^{μ}_{α} in the conventional frame and lab frames, respectively. But the components in the conventional frame are just $e'^{\mu}_{\alpha} = \delta^{\mu}_{\alpha}$ (the components of any set of basis vectors in their own frame), so Eq. (5.8) gives

$$e^{\mu}_{\alpha} = L^{\mu}{}_{\alpha}(-\mathbf{v}). \tag{5.10}$$

Now using Eqs. (5.9) to read off the components of L^{-1}, we find

$$e^0_0 = \gamma, \qquad e^i_0 = \gamma v_i, \tag{5.11a}$$

$$e^0_j = \gamma v_j, \qquad e^i_j = \delta_{ij} + (\gamma - 1)\frac{v_i v_j}{v^2}, \tag{5.11b}$$

where $i, j = 1, 2, 3$ and where v_i are the components of the velocity of the particle in the lab frame. Later we will use these basis vectors of the Thomas conventional frame to compute Thomas's angular velocity.

Now we turn to a physical problem that has an elegant solution in terms of the geometry of space-time. Imagine a gyroscope carried along by a relativisitic spaceship. The gyroscope is anchored to the spaceship through its center of mass, and otherwise is spinning freely. Thus, there are no torques on the gyroscope, and it should continue to point in the same direction as the spaceship accelerates and moves through space.

Indeed, in Newtonian (pre-relativistic) mechanics, this is precisely what happens. The gyroscope on the rocket points always in the same direction as seen either by an inertial frame or by an accelerated but rotationless frame carried by the rocket, where a rotationless frame is one whose axes are always parallel to the axes of an inertial frame. But in relativistic mechanics, the axes of a moving frame cannot in general be parallel to the axes of a stationary frame. You will be misled on this point if you think only of boosts down the x, y, or z axes, because in those special cases, the axes of the moving frame are parallel to the axes of the stationary frame (there is just a Lorentz contraction in the direction of motion). But for boosts in an arbitrary direction, the axes of the moving frame, as seen in the stationary frame, are not even orthogonal (they tend to flatten up into the plane orthogonal to $\mathbf{v}$, according to the "pancake" effect).

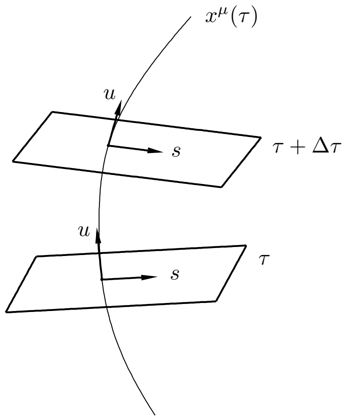

Fig. 5.2. The spin or angular momentum vector s of a gyroscope is a vector that is purely spatial in any rest frame of the particle, that is, it is orthogonal to the world velocity u.

So what does "in the same direction" mean in a relativistic context? As the spaceship follows its world line $x^\mu = x^\mu(\tau)$, at any given value of τ the angular momentum or spin $\mathbf{s}$ of the gyroscope is a vector that is purely spatial in some (hence any) instantaneous rest frame of the spaceship. Such a vector can be regarded as a 4-vector whose time component vanishes in a rest frame, that is, one that is orthogonal to the world velocity u, as indicated in Fig. 5.2. Thus we define a 4-vector with components

$$s^\mu = \begin{pmatrix} 0 \\ \mathbf{s} \end{pmatrix} \tag{5.12}$$

in some conventional rest frame, and then define the components in all other Lorentz frames by doing Lorentz transformations. The result is a 4-vector (a set of quantities that transform as a 4-vector). In a rest frame, the components of u are given by

$$u^{\mu} = \begin{pmatrix} 1 \\ \mathbf{0} \end{pmatrix}, \tag{5.13}$$

so we have

$$s^{\mu} u_{\mu} = (s \cdot u) = 0, \tag{5.14}$$

a condition that is covariant hence valid in any Lorentz frame. When we do a Lorentz transformation to some other frame (for example, the lab frame), s^{μ} will acquire a nonzero time component and u^{μ} will acquire nonzero spatial components, but Eq. (5.14) still holds.

Now obviously there must be some physics that determines the evolution of s along the world line of the particle. If we are in the space ship observing the gyroscope, it will point in whatever direction it wants to, and we can measure the components of the vector $\mathbf{s}$ with respect to any conventional rest frame we choose. But what physical law determines the direction $\mathbf{s}$ points in?

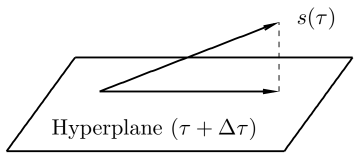

Fig. 5.3. The spin vector s at proper time τ shown in relation to the three-dimensional hyperplane at proper time $\tau + \Delta\tau$.

The answer is indicated in Fig. 5.3, which illustrates the three-dimensional space-like hyperplane orthogonal to u at proper time $\tau + \Delta\tau$, and the spin vector s at proper time τ. Because the world velocity u has changed between τ and $\tau + \Delta\tau$, the three-dimensional hyperplane at proper time τ, in which $s(\tau)$ lies, is not the same as the three-dimensional hyperplane at time $\tau + \Delta\tau$. Thus, $s(\tau)$ does not lie in the hyperplane at $\tau + \Delta\tau$, it has components both parallel and perpendicular to this hyperplane, as shown in the figure. Now the question is, what is $s(\tau + \Delta\tau)$? It must lie in the hyperplane at $\tau + \Delta\tau$. If it is not the projection of $s(\tau)$ onto this hyperplane, then it must be some some other vector related to this projection by a spatial rotation, that is, a rotation in the hyperplane itself. But there is nothing to indicate any direction into which the vector could be rotated, that is, no physics to indicate any direction of a rotation. Therefore the $s(\tau + \Delta\tau)$ must be the projection of $s(\tau)$ onto the hyperplane at $\tau + \Delta\tau$, at least in the limit $\Delta\tau \to 0$.

In this limit, the projection has the same length as the original vector $s(\tau)$, since the increment is an infinitesimal vector orthogonal to the hyperplane. This increment is parallel to u, so the equation of evolution of the vector s must be

$$\frac{ds}{d\tau} = ku, \tag{5.15}$$

where k is a proportionality factor. This factor can be determined by differentiating Eq. (5.14),

$$0 = \frac{d(s\cdot u)}{d\tau} = (\frac{ds}{d\tau}\cdot u) + (s\cdot\frac{du}{d\tau}) = k + (s\cdot b), \tag{5.16}$$

or

$$\boxed{\frac{ds}{d\tau} = -(s\cdot b)u.} \tag{5.17}$$

This is the desired equation of evolution for the spin s. Assuming that $x^\mu(\tau)$ is given, then $u = dx/d\tau$ and $b = du/d\tau$ are known functions of τ, and Eq. (5.17) is a linear differential equation for s, which can be solved once some initial condition s_0 is given at $\tau = \tau_0$. This equation is called a *parallel transport* equation, of which there are many kinds in physics. This particular one is called *Fermi-Walker transport*. Fermi-Walker transport is the way of transporting a purely spatial vector along the world line of a particle in such a way that it is as “rotationless” as possible, given that it must remain orthogonal to the world velocity u. The evolution takes place through a sequence of infinitesimal boosts that transform $s(\tau)$ into $s(\tau+\Delta\tau)$. These infinitesimal Lorentz transformations are pure boosts, that is, they do not contain any rotation.

Now consider the components of the spin $\mathbf{s}$ of the gyroscope as measured with respect to some conventional rest frame. This need not be the Thomas conventional rest frame, for example in the spaceship fantasy we might think of some orthonormal frame rigidly attached to the spaceship. Then the components of $\mathbf{s}$ with respect to this conventional frame are not necessarily constant in time, because the basis vectors of the conventional frame are not necessarily “rotationless.” That is, these vectors do not necessarily satisfy Eq. (5.17).

On the other hand, we can create a “rotationless” rest frame by setting up three gyroscopes with orthogonal spins, and use these to define the spatial unit vectors of a “rotationless” or parallel-transported frame. Let these vectors be f_i, $i = 1, 2, 3$. If these vectors form an orthonormal frame at some initial time τ_0, then Fermi-Walker transport guarantees that they form an orthonormal frame at all subsequent times. This is because

Fermi-Walker transport preserves scalar products, that is, if p and q are two purely spatial vectors in the rest frame ($(p \cdot u) = (q \cdot u) = 0$) that satisfy

$$\frac{dp}{d\tau} = -(p \cdot b)u, \qquad \frac{dq}{d\tau} = -(q \cdot b)u, \tag{5.18}$$

then

$$\frac{d(p \cdot q)}{d\tau} = (\frac{dp}{d\tau} \cdot q) + (p \cdot \frac{dq}{d\tau}) = -(p \cdot b)(q \cdot u) - (p \cdot u)(q \cdot b) = 0. \tag{5.19}$$

Thus, if $(f_i \cdot f_j) = g_{ij} = -\delta_{ij}$ for $i, j = 1, 2, 3$ at the initial proper time, this condition holds for all proper time. Then, defining $f_0 = u$, we have an orthonormal tetrad $\{f_\alpha\}$ for a rotationless rest frame all along the world line of the particle.

Equation (5.19) has another consequence, for if s is a spin that is parallel transported, then its spatial components with respect to the parallel transported frame are constant, since

$$\frac{d(s \cdot f_i)}{d\tau} = 0 \tag{5.20}$$

(see Eq. (5.5)). This just means that a parallel transported vector such as $\mathbf{s}$ is fixed in the parallel transported frame, an obvious conclusion. Yet another consequence is

$$\frac{d(s \cdot s)}{d\tau} = 0, \tag{5.21}$$

which just says that the length of the spin $\mathbf{s}$ is preserved by Fermi-Walker transport (as previously noted).

The conventional frame $\{e_\alpha\}$ and the parallel transported frame $\{f_\alpha\}$ can be given a vivid physical interpretation. If we are riding on the spaceship and fix ourselves in some conventional frame (perhaps one rigidly attached to the spaceship), then in general we will feel centrifugal forces in addition to the forces of acceleration. But if we fix ourselves to the parallel transported frame, we will feel no centrifugal forces. In this direct physical sense, we must say that the parallel transported is the rotationless frame, while the conventional frame in general is rotating.

As we have noted, the spin vector $\mathbf{s}$ is fixed as viewed from the parallel transported frame, but it is not fixed as viewed from some conventional frame. That is, the components of $\mathbf{s}$ with respect to the frame $\{e_\alpha\}$ are in general functions of time. Let us consider the contravariant components s^i, $i = 1, 2, 3$ of the vector s in the conventional rest frame, which, according to Eq. (5.12) are the usual Cartesian components of $\mathbf{s}$ with respect to the spatial unit vectors of this frame. Since $s^\mu = g^{\mu\alpha}(s \cdot e_\alpha)$, we have $s^i = -(e_i \cdot s)$, or

$$\frac{ds^i}{d\tau} = -(\frac{de_i}{d\tau} \cdot s) - (e_i \cdot \frac{ds}{d\tau}). \tag{5.22}$$

But the second term of this equation vanishes, since s satisfies Eq. (5.17) and $(e_i \cdot u) = 0$. As for the first term, we can write $s = s^\nu e_\nu = s^k e_k$ since $s^0 = 0$, or

$$\frac{ds^i}{d\tau} = -(\frac{de_i}{d\tau} \cdot e_k) s^k = -\Omega_{ik}\, s^k, \tag{5.23}$$

where

$$\Omega_{ik} = (\frac{de_i}{d\tau} \cdot e_k). \tag{5.24}$$

This tensor is antisymmetric, as we see if we differentiate $(e_i \cdot e_k) = -\delta_{ij}$ with respect to τ:

$$(\frac{de_i}{d\tau} \cdot e_k) + (e_i \cdot \frac{de_k}{d\tau}) = 0. \tag{5.25}$$

Therefore Ω_{ij} can be written in terms of a vector $\boldsymbol{\omega}$,

$$\Omega_{ij} = \epsilon_{ijk}\, \omega_k, \tag{5.26}$$

where we have used Eq. (3.10) (in the notes on the Levi-Civita symbol). In terms of this vector, the Eq. (5.23) becomes

$$\frac{ds^i}{d\tau} = \epsilon_{ijk}\, \omega_j s^k, \tag{5.27}$$

or,

$$\frac{d\mathbf{s}}{d\tau} = \boldsymbol{\omega}\times\mathbf{s}. \tag{5.28}$$

The vector $\boldsymbol{\omega}$ represents the rate of rotation of the parallel transported frame as seen from the conventional rest frame, and hence the rate of rotation of any vector fixed in the parallel transported frame. It is given explicitly by

$$\omega_i = \frac{1}{2}\epsilon_{ijk}(\frac{de_j}{d\tau} \cdot e_k). \tag{5.29}$$

It would be more logical to speak of the rate of rotation of the conventional rest frame as seen from the parallel transported frame, since the former is really rotating and the latter is not, but this is how we will do it.

If the Thomas conventional frame is used, in which the basis vectors $\{e_i\}$ have the components given by Eqs. (5.11), then the angular velocity computed in Eq. (5.29) is Thomas's angular velocity $\boldsymbol{\omega}_T$. The calculation requires some algebra, which is relatively uninteresting. The algebra is simplified by noting that only the antisymmetric part of $((de_j/d\tau) \cdot e_k)$ enters into the computation of $\boldsymbol{\omega}$, so that we can throw away any terms that are symmetric in (jk) that arise in the computation. The result is

$$\boldsymbol{\omega}_T = \frac{\gamma^3}{\gamma+1}\frac{\mathbf{a}\times\mathbf{v}}{c^2}, \tag{5.30}$$

where $\mathbf{a} = d\mathbf{v}/dt$ is the 3-acceleration of the particle as seen in the lab frame and where we have restored factors of c. This is the same as Jackson's Eq. (11.119), except for an extra factor of γ, which is due to the fact that here we are measuring the angular velocity in the rest frame of the particle (with respect to proper time), not the lab frame, as Jackson does. It is clear that this angular velocity is really native to the rest frame of the particle, since it measures the rate of rotation between the conventional and parallel transported rest frames.

You may worry about the positioning of the indices (upper or lower) in formulas like Eqs. (5.11), (5.23) or (5.24). In general the upper or lower position of an index makes a statement about how the object transforms under coordinate transformations. Starting with Eq. (5.22), however, we specialized to one frame, the conventional rest frame, and stopped trying to make covariant statements that would be valid in any frame. We maintained the upper position on the components s^i of the spin vector, because these actually are the contravariant components of s^μ (with $\mu = i$) in the conventional rest frame, but we just used lower (subscript) indices on $\boldsymbol{\omega}$ because it is a 3-vector, measured in one frame only (the conventional rest frame) and we never attempt to make it a part of any 4-vector. Similarly, the notation v_i in Eq. (5.11) refers to the components of the velocity 3-vector $d\mathbf{x}/dt$ of the particle in the lab frame. This is never converted to any other frame, nor is it ever interpreted as a part of a 4-vector, so we just use lower indices on it. Notice that Thomas's formula, Eq. (5.30), is a hybrid expression, since $\boldsymbol{\omega}$ is measured in the conventional rest frame, and $\mathbf{v}$ and $\mathbf{a}$ are measured in the lab frame.

Thomas precession is actually a gauge theory, associated with the arbitrariness in the choice of the conventional rest frame. We will not attempt to define what a gauge theory is in general, but there is another gauge theory we are familiar with, associated with the arbitrariness in the choice of the potentials Φ and $\mathbf{A}$ for given electric and magnetic fields, and it is worthwhile pointing out some similarities. In the case of the electromagnetic field, we can say that the potentials contain both a physical part (since $\mathbf{E}$ and $\mathbf{B}$ can be computed from the potentials) and a nonphysical part, which can be established only by convention. A gauge transformation changes the nonphysical part. Thus, we can say that the potentials provide a mathematically redundant specification of a single physical reality. Similarly, in Thomas precession, the physical reality, determined by the particle itself and independent of observer, is the world velocity u. When we set up coordinates in the 3-dimensional space-like hyperplane orthogonal to u, however, there is a large degree of arbitrariness in the choice of the spatial unit vectors $\{e_i\}$. These vectors are not completely arbitrary (they contain a physical element), since they are required to be orthogonal to u, and, in fact, taken together they specify u. But they also contain a nonphysical element, which is their arbitrary orientation in the 3-space that they span. A change in this orientation is specified by an orthogonal matrix $R(\tau)$ as in Eq. (5.6). This is a gauge transformation in this theory.

Whenever we have a theory containing an arbitrary convention, we should examine what parts of the theory depend on the choice of that convention and what parts do not. The former we call gauge-dependent, and the latter, gauge-invariant. Clearly, physically meaningful results must be gauge-invariant. In the case of Thomas precession, the parallel transported frame is gauge-invariant, but the conventional frame is gauge-dependent. Therefore the angular velocity $\boldsymbol{\omega}_T$ given by Eq. (5.30) is gauge-dependent, since it is the rate of rotation of the parallel transported frame with respect to the Thomas conventional rest frame. To emphasize this point, we may consider another observer with lab basis vectors $\{\ell'_\alpha\}$, as discussed above, and corresponding Thomas conventional frame $\{e'_\alpha\}$. There is a different angular velocity $\boldsymbol{\omega}'_T$ connecting the frame $\{e'_\alpha\}$ with the the parallel transported frame. The two angular velocities $\boldsymbol{\omega}_T$ and $\boldsymbol{\omega}'_T$ are not just the components of one vector as viewed in two different frames. (You might think that they are somehow the spatial parts of a 4-vector, measured in the two frames $\{\ell_\alpha\}$ and $\{\ell'_\alpha\}$, but this is not the case.) This is because they represent different things geometrically, since there actually are different rates of rotation of the parallel transported frame with respect to the two conventional frames.

Since Thomas's angular velocity is gauge-dependent, we must ask, why are we interested in it, and why does it give the correct spin-orbit interaction in atomic physics? To answer this, consider an accelerated particle in a periodic orbit, so that $\mathbf{x}(t)$, $\mathbf{v}(t)$ and $u^\mu(\tau)$ are all periodic. (The period with respect to proper time τ is not the same as the period with respect to coordinate time t, but there is periodicity with respect to either variable.) Thomas's conventional rest frame depends only on $\mathbf{v}$, so after a period of the motion when $\mathbf{v}$ returns to its original value, so also does Thomas's conventional frame. The world velocity u also returns to its original value, which means that the 3-dimensional space-like hyperplane orthogonal to u returns to its original value. Thus, vectors in the original hyperplane can be compared to vectors in the final hyperplane, since the two hyperplanes are identical (or parallel). In particular, we can examine vectors that have been Fermi-Walker transported along the world line of the particle over a period of the motion, and compare them to their original values. If we do this with the parallel transported frame, we find in general that the parallel transported frame after a period of the motion is related to the original frame by some rotation. This is in spite of the fact that the parallel transported frame is "rotationless," without centrifugal forces.

On the other hand, no such comparison can be made over an interval of time that is not an integer multiple of a period, because in that case the initial and final hyperplanes are not the same. It is possible to ask how much the parallel transported frame has rotated relative to some conventional frame, even after a fraction of a period, but since the conventional frame is conventional, the answer is gauge-dependent. However, after an integer multiple of a period, the conventional frame has returned to its orginal value, and the amount of rotation of the parallel transported frame with respect to the conventional frame is now gauge-invariant. This rotation is one that the parallel transported frame accumulates over and over in the periodic motion.

Thus, over a long period of time, Thomas's angular velocity $\boldsymbol{\omega}_T$ does give the average rate of rotation of the parallel-transported frame, which is a gauge-invariant quantity. Taking expectation values in quantum mechanics is equivalent to performing a long-time average, and I suspect this is why Thomas's calculation is able to give the correct spin-orbit splitting. It is not worth it to pursue this question too far, since the proper way to treat spin-orbit effects is to use the Dirac equation, which in a sense has Thomas precession built into it automatically. But a proper understanding of Thomas precession requires one to treat it as a gauge theory. None of this was appreciated in Thomas's time, however.

Thomas precession is closely related to Berry's phase, another example of a gauge theory.