Physics 209
Fall 2002
Notes 3
The Levi-Civita Symbol

The Levi-Civita symbol is useful for converting cross products and curls into the language of tensor analysis, and for many other purposes. The following is a summary of its most useful properties in three-dimensional Euclidean space.

The Levi-Civita symbol is defined by

$$\epsilon_{ijk} = \begin{cases} 1, & \text{if } (ijk) \text{ is an even permutation of } (123); \\ -1, & \text{if } (ijk) \text{ is an odd permutation of } (123); \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

It has 27 components, of which only 6 are nonzero. It follows directly from this definition that ϵ_{ijk} changes sign if any two of its indices are exchanged,

$$\epsilon_{ijk} = \epsilon_{jki} = \epsilon_{kij} = -\epsilon_{jik} = -\epsilon_{ikj} = -\epsilon_{kji}. \tag{3.2}$$

The Levi-Civita symbol is convenient for expressing cross products and curls in tensor notation. For example, if **A** and **B** are two vectors, then

$$(\mathbf{A}\times\mathbf{B})_i = \epsilon_{ijk}\, A_j B_k, \tag{3.3}$$

and

$$(\nabla\times\mathbf{B})_i = \epsilon_{ijk}\, \frac{\partial B_k}{\partial x_j}. \tag{3.4}$$

Any combination of an even number of Levi-Civita symbols (or an even number of cross products and curls) can be reduced to dot products with the following system of identities. Similarly, any combination of an odd number of Levi-Civita symbols (or an odd number of cross products and curls) can be reduced to a single Levi-Civita symbol (or a cross product or a curl) plus dot products. The first is the most general:

$$\epsilon_{ijk}\,\epsilon_{\ell mn} = \begin{vmatrix} \delta_{i\ell} & \delta_{im} & \delta_{in} \\ \delta_{j\ell} & \delta_{jm} & \delta_{jn} \\ \delta_{k\ell} & \delta_{km} & \delta_{kn} \end{vmatrix}. \tag{3.5}$$

Notice that the indices (ijk) label the rows, while (ℓmn) label the columns. If this is contracted on i and l, we obtain

$$\epsilon_{ijk}\,\epsilon_{imn} = \begin{vmatrix} \delta_{jm} & \delta_{jn} \\ \delta_{km} & \delta_{kn} \end{vmatrix} = \delta_{jm}\,\delta_{kn} - \delta_{jn}\,\delta_{km}. \tag{3.6}$$

This identity is the one used most often, for boiling down two cross products that have one index in common, such as $\nabla\times(\mathbf{A}\times\mathbf{B})$. By contracting Eq. (3.6) in j and m we obtain

$$\epsilon_{ijk}\,\epsilon_{ijn} = 2\delta_{kn}. \tag{3.7}$$

Finally, contracting on k and n we obtain

$$\epsilon_{ijk}\,\epsilon_{ijk} = 6. \tag{3.8}$$

It should be clear how to generalize these identities to higher dimensions.

If $A_{ij} = -A_{ji}$ is an antisymmetric, 3×3 tensor, it has 3 independent components that we can associate with a 3-vector $\mathbf{A}$, as follows:

$$A_{ij} = \begin{pmatrix} 0 & A_3 & -A_2 \\ -A_3 & 0 & A_1 \\ A_2 & -A_1 & 0 \end{pmatrix} = \epsilon_{ijk}\,A_k. \tag{3.9}$$

The inverse of this is

$$A_{ij} = \frac{1}{2}\epsilon_{ijk}\,A_k. \tag{3.10}$$

Using thes identities, the multiplication of an antisymmetric matrix times a vector can be reexpressed in terms of a cross product. That is, if

$$X_i = A_{ij}\,Y_j \tag{3.11}$$

then

$$\mathbf{X} = \mathbf{Y}\times\mathbf{A}. \tag{3.12}$$

Similarly, if $\mathbf{A}$ and $\mathbf{B}$ are two vectors, then

$$A_iB_j - A_jB_i = \epsilon_{ijk}\,(\mathbf{A}\times\mathbf{B})_k, \tag{3.13}$$

and

$$\frac{\partial B_j}{\partial x_i} - \frac{\partial B_i}{\partial x_j} = \epsilon_{ijk}\,(\nabla\times\mathbf{B})_k. \tag{3.14}$$

Finally, if M_{ij} is a 3×3 matrix (or tensor), then

$$\det M = \epsilon_{ijk}\,M_{1i}M_{2j}M_{3k} = \frac{1}{6}\epsilon_{ijk}\,\epsilon_{\ell mn}\,M_{i\ell}M_{jm}M_{kn}. \tag{3.15}$$

The Levi-Civita symbol has been defined here only on $\mathbb{R}^3$, but most of the properties above are easily generalized to $\mathbb{R}^n$ (including the case $n = 2$). It only transforms as a tensor under proper orthogonal changes of coordinates, which is why we are calling it a "symbol" instead of a "tensor." It can, however, be used to create so-called *tensor densities* on arbitrary manifolds with a metric, and has fascinating applications in Hodge-de Rham theory in differential geometry.