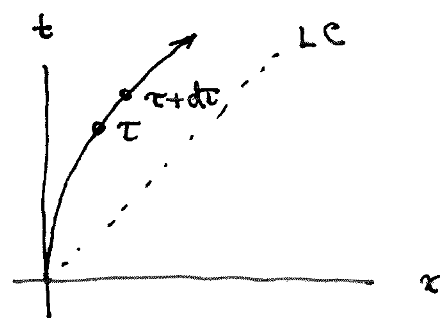
5.22 Relativistic Rocket.. (Special Relativity)

2 ways to do it: In rest frame, or in lab frame.

I. Lab frame.

World line of rocket: asymptotes to 45°.



Let $m(\tau)$ be mass of rocket as fn of τ.
Sim $p(\tau)$ " momentum in lab frame as fn of τ
$E(\tau)$ " energy " " " " " " "
$v(\tau)$ " velocity " " " " " " "
$\gamma(\tau)$ " $\frac{1}{\sqrt{1-v^2}}$ " " " " " " "

Note $\gamma \equiv \frac{1}{\sqrt{1-v^2}}$, $E = \gamma m$, $p = \gamma m v$ (3 relations among 5 vars)

So among p, E, v, γ, m there are really only 2 indep. vars, say, m, v.

Important Fact! Rest mass is not conserved!!

Energy and momentum are the only two conservation laws!

(In nonrelativistic mechanics, rest mass is conserved, eg. If in time dt the rocket ejects small mass ~~dm~~ out the exhaust, then mass of exhaust ejected in time dt + mass of rocket at time t+dt = mass of rocket at time t. But this is not true relativistically.)

Let $dm = m(\tau+d\tau) - m(\tau)$
$dp = p(\tau+d\tau) - p(\tau)$
$dE = E(\tau+d\tau) - E(\tau)$
$dv = v(\tau+d\tau) - v(\tau)$
$d\gamma = \gamma(\tau+d\tau) - \gamma(\tau)$

Note, $dm < 0$, since mass of rocket is decreasing.

Since we have overall conservation of energy and momentum, let's look at E, p first.

Is $dE > 0$ or < 0? Rocket keeps going faster, $dv > 0$, but mass decreases, $dm < 0$. ?

$$\text{Energy of rocket at time } \tau + d\tau = \underbrace{E(\tau)}_{''} + dE$$
$$\text{'' '' '' '' '' } \tau = E(\tau)$$

So, energy of exhaust emitted in time $d\tau$ must be $-dE$.

Since energy of exhaust must be > 0, we see $dE < 0$. (answer to question above.)

Similarly,

$$\text{momentum of rocket at time } \tau + d\tau = \underbrace{p(\tau)}_{''} + dp$$
$$\text{'' '' '' '' '' } \tau = p(\tau)$$

So ~~energy~~ momentum of exhaust emitted in time $d\tau = -dp$.
Note, we don't know whether $dp > 0$ or $dp < 0$.

So we know the energy and momentum of exhaust emitted in time $d\tau$, they are $-dE$, $-dp$ respectively.

Digression: Useful fact in special relativity:

For any body of mass m, energy E, momentum p etc,

$$\left.\begin{aligned} E &= m\gamma \\ p &= m\gamma v \end{aligned}\right\} \Rightarrow v = \frac{p}{E}.$$

Useful way to get velocity from E, p.

Now apply this digression to the element of exhaust ejected by rocket in time $d\tau$:

$$v_{exh.} = \frac{p_{exh.}}{E_{exh.}} = \left(\frac{-dp}{-dE}\right)_{rocket} = \left(\frac{dp}{dE}\right)_{rocket}.$$

This is all in <u>lab frame</u>!

But we know that the velocity of the exhaust <u>relative</u> to the rocket is u (a given, fixed number). And, of course, the ~~velocity~~ exhaust is ejected in the $-x$ direction.

So we use rule for addition of velocities. In this case,

$$\underset{\substack{\uparrow \\ \text{lab.}}}{v_{exh}} = \frac{v-u}{1-uv} \qquad v = \text{vel. of rocket in lab frame.}$$

So we get the equation, (for rocket)

$$\boxed{\frac{dp}{dE} = \frac{v-u}{1-uv}.} \qquad \begin{array}{l} u = \text{const}, \\ p, E, v = \text{fns. of } \tau \end{array}$$

Let's eliminate p, E in favor of v.

$$p = m\gamma v = Ev.$$

$$E = m\gamma$$

$$dE = m\,d\gamma + \gamma\,dm \qquad \frac{dp}{dE} = \frac{v\,dE + E\,dv}{dE}$$

$$dp = v\,dE + E\,dv.$$

$$= v + E\frac{dv}{dE} = \frac{v-u}{1-uv}.$$

$$E\frac{dv}{dE} = \frac{v-u}{1-uv} - v = -\frac{u(1-v^2)}{1-uv} = -\frac{u}{(1-uv)\gamma^2}$$

$$E\,\frac{dv}{dE} = -\frac{u}{(1-uv)\gamma^2}$$

Cross multiply, use $dE = m\,d\gamma + \gamma\,dm$, $E = m\gamma$:

$$+ \quad v\,dv\,\gamma^3$$

$$(m\gamma\,dv)(1-uv)\gamma^2 = -u\,dE = -u\,(m\,d\gamma + \gamma\,dm)$$

$$m\gamma^3 dv - m\gamma^3 uv\,dv = -uv\,m\gamma^3 dv - u\gamma\,dm$$

cancel.

$$m\gamma^2 dv = -u\,dm$$

$$\frac{1}{u}\,\frac{dv}{1-v^2} = -\frac{dm}{m}.$$

$$\frac{1}{2u}\ln\left(\frac{1+v}{1-v}\right) = -\ln m + \text{const.}$$

Let $v = 0$, $m = m_0$ initially.

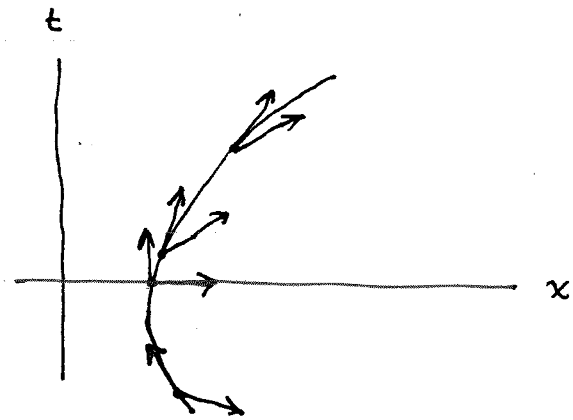
$$\Rightarrow \quad \ln\frac{m}{m_0} = \frac{1}{2u}\ln\left(\frac{1-v}{1+v}\right)$$

$$\boxed{m = m_0\left(\frac{1-v}{1+v}\right)^{1/2u}}$$

II. In rest frame of rocket.

Algebra easier, conceptually a little more difficult.

Realize that an accelerated particle has more than one rest frame, in fact there is a different rest frame at each value of τ.



Work in rest frame at proper time τ. Then: (parameters of rocket).

	at time τ
mass	$m(\tau)$
energy	$m(\tau)$
momentum	0
velocity	0
γ	1

Now look at params of rocket at time $\tau + d\tau$, as measured in rest frame at time τ:

	~~at time $\tau+d\tau$~~
mass	$m(\tau+d\tau) = m(\tau) + dm$
energy	
momentum	$\cancel{m(\tau)\gamma(\tau)}$ $m(\tau+d\tau)\,\gamma(\tau+d\tau)\,v(\tau+d\tau) =$
velocity	dv
γ	$\frac{1}{\sqrt{1-(dv)^2}} = 1 + \frac{1}{2}(dv)^2 = 1$ to 1st order.

Velocity: $v(\tau+d\tau) = v(\tau) + dv = dv$

γ: $\gamma(\tau+d\tau) = \dfrac{1}{\sqrt{1-v(\tau+d\tau)^2}} = \dfrac{1}{\sqrt{1-(dv)^2}} = 1+\frac{1}{2}dv^2 = 1$

through 1st order.

E: $E(\tau+d\tau) = m(\tau+d\tau)\,\gamma(\tau+d\tau)$

$= (m+dm)(1) = m+dm$

p: $p(\tau+d\tau) = m(\tau+d\tau)\,\gamma(\tau+d\tau)\,v(\tau+d\tau)$

$\doteq (m+dm)(1)(dv) = m\,dv$ thru 1st order.

So, dE, dp of rocket given by:

$dE = dm$

$dp = m\,dv$.

So, by conservation of energy, momentum, the energy, momentum of exhaust emitted in time $d\tau$ are

$E_{exh.} = -dE = -dm$

$p_{exh} = -dp = -m\,dv$

So, $v_{exh} = \dfrac{p_{exh}}{E_{exh}} = \left(\dfrac{dE}{dp}\right)_{rocket} = \dfrac{dm}{m\,dv}$.

But $v_{exh} = -u$ in rest frame, so,

$\dfrac{dm}{m\,dv} = -u$, $\quad \dfrac{dm}{m} = -u\,dv$.

But this dv is the change in velocity of rocket as seen in rest frame at time τ. To get $(dv)_{lab}$, we use addition of velocities again,

$$(dv)_{lab} = \frac{\overset{(v_{lab})}{v} + (dv)_{rest}}{1 + v(dv)_{rest}} - v = (v + dv_{rest})(1 - v\,dv_{rest}) - v$$

$$= v + dv_{rest}(1 - v^2) - v$$

$$= dv_{rest}(1 - v^2). \qquad (\text{Here } v = v_{lab}.)$$

So, $$\frac{dm}{m} = -u\,(dv)_{rest} = \;= -\frac{u}{1-v^2}(dv)_{lab}.$$

Drop lab subscript,

$$\frac{dm}{m} = -u\,\frac{dv}{1-v^2}.$$

Same differential equation as before.