12.16. Seeing the surface of the star fade to oblivion as it approaches the event horizon. The "seeing" is done by an observer at a large distance $R \gg M$, who receives light signals sent back by ~~the~~ a particle (say) on the surface of the collapsing star.

For simplicity we assume the star is made of "dust", i.e. particles with zero pressure. Each dust particle falls in the gravitational field (its world line is a geodesic), and there are no collisions with other dust particles. We do not know what the metric is in the inside of the star, so we cannot compute orbits there, but outside the metric is Schwarzschild. A particle at the surface of the star is at the edge of the vacuum region, so it follows a geodesic in Schwarzschild geometry. For simplicity we assume the whole star collapses radially inward; there is no motion in θ or ϕ, only in r. also, for simplicity, we assume the particle at the surface starts at large r (effectively, $r = \infty$) with zero velocity. The orbit is a radial plunge orbit.

As the particle falls in, it emits photons of frequency ω_* (fixed), as seen by the particle. These photons are received by the stationary observer at radius R.

This problem involves 3 orbits in Schwarzschild geometry: 1) the infalling particle; 2) the photon; 3) the stationary observer at ∞.

~~As the~~ When the observer at ∞ (radius $R \gg M$) receives a photon, it has frequency ω_∞. As successive photons are received from the infalling particle, they are progressively red shifted, so ω_∞ decreases. The problem is to derive a formula for $\frac{\omega_\infty}{\omega_*}$ as a function of t_R, which is the time at which the observer at radius $R \gg M$ receives the photon. The book wants you to compare your answer to (12.11), which says

$$\frac{\omega_\infty}{\omega_*}(t_R) \propto e^{-t_R/4M}.$$

First draw a space-time diagram. I won't attempt it here, see Fig. 12.3 in the book. When the surface of the star is at large radius, the photon orbits are nearly 45° outgoing straight lines, because at large r the Schwarzschild geometry becomes essentially Minkowski. As r decreases toward $r=2M$, the initial slope of the photon orbits points more toward the vertical, indicating the tilting of the light cones. (See Fig 12.2 for that.) As a result, the photon arrive at the distant observer later and later.

Diagrams 12.2 and 12.3 actually use Finkelstein-Eddington coordinates $\bar{t}, r$ instead of Schwarzschild coordinates t, r, because the latter are singular at $r=2M$. This problem only concerns behavior for $r > 2M$ so we can use Schwarzschild coordinates for the analysis. We use the diagrams 12.2 and 12.3 only for their qualitative features, and we do not need to worry about F-E coordinates. The point is that Fig 12.3 shows clearly that the time interval between photons received at ∞ ($r = R \gg M$) increases as time goes on.

Suppose the photons are emitted at equal time intervals $\Delta\tau$ as seen by a clock carried by the infalling particle.

Then the Δt of ~~the~~ between the times the photons are received increases.

Actually, the ratio $\frac{\Delta t}{\Delta \tau}$ is closely related to the ratio $\frac{\omega_\infty}{\omega_*}$ that we want to compute. The infalling particle does not have to emit photons at equal intervals $\Delta\tau$, it can be any periodic signal. Suppose it is a continuous light wave of frequency ω_*, and let $\Delta\tau = \frac{2\pi}{\omega_*}$ = time between wave crests emitted. Then $\Delta t = \frac{2\pi}{\omega_\infty}$ = time between wave crests received. Then

$$\frac{\Delta t}{\Delta \tau} \to \frac{dt}{d\tau} = \frac{\omega_*}{\omega_\infty}.$$

Here we can think of a function $t = t(\tau)$, giving the time that a photon (or wave crest) is received as a function of the proper time along the orbit of the infalling particle. The ratio $\frac{\omega_*}{\omega_\infty}$ is the derivative of this function.

We need to look at 3 orbits, particle (infalling), photon, distant observer.

<u>First</u>, the <u>orbit of the infalling particle</u>. The book treats general orbits in Schwarzschild geometry, and this can lead to some complicated formulas. But this infalling orbit is a "radial plunge" orbit so θ and ϕ don't enter, and the analysis is easy to do with a "desert island" method.

We use 2 facts to analyze this orbit: 1st, the conservation law that comes from the ignorable coordinate t; 2nd the requirement $\underline{u} \cdot \underline{u} = -1$. (Plus the fact that it's radial plunge).

Since the orbit is radial plunge,

$$\underset{\sim}{u} = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}, 0, 0 \right) \qquad (\text{coords } (t\, r\, \theta\, \phi)).$$

Because of ~~to~~ t-translation invariance, we have the Killing vector $\underset{\sim}{\xi} = (1, 0, 0, 0)$ and hence the conserved qty

$$\underset{\sim}{\xi} \cdot \underset{\sim}{u} = g_{tt}\, \xi^t u^t = -\left(1 - \frac{2M}{r}\right)(1)\left(\frac{dt}{d\tau}\right) = \text{const.} \equiv -e,$$

or

$$\boxed{\left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} = e.}$$

What is e? We are assume the infalling particle started at $r = \infty$ with zero velocity, ~~so~~ so evaluate the above expression there and we get $e = 1$. So for this particle,

$$\frac{dt}{d\tau} = \frac{1}{1 - \frac{2M}{r}}.$$

This is one of the 2 components of $\underset{\sim}{u}$. To get $\frac{dr}{d\tau}$, we use

$$-1 = \underset{\sim}{u} \cdot \underset{\sim}{u} = -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \frac{1}{1 - \frac{2M}{r}}\left(\frac{dr}{d\tau}\right)^2$$

or, doing the algebra,

$$\left(\frac{dr}{d\tau}\right)^2 = \frac{2M}{r}.$$

We take − sign on square root, since the particle is infalling, and we get

$$\frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}}.$$

Altogether, we have

$$\underset{\sim}{u} = \left(\frac{1}{1-\frac{2M}{r}},\ -\sqrt{\frac{2M}{r}},\ 0,\ 0 \right)$$

$$= \left(\frac{dt}{d\tau},\ \frac{dr}{d\tau},\ 0,\ 0 \right).$$

for the infalling particle.

We can use what we have so far to solve some differential equations to get $r(\tau)$ and $t(\tau)$ for the infalling particle, but let's postpone that as long as possible, until we need them, since it leads to complicated formulas.

Turn to <u>second orbit</u>, that of photon. The photon is emitted at event (r, t) on world line of infalling particle, parameterized by τ, the proper time of the infalling particle, and it is received at event (R, t_R) by observer at ∞.

The photon equations of motion are the same as those of a massive particle, except τ gets replaced by the affine parameter λ. With a proper choice of λ, $\underset{\sim}{u} = \frac{d\underset{\sim}{x}}{d\lambda}$ is physically the momentum 4-vector of the photon (confusingly, this would be $m\underset{\sim}{u} = m\frac{d\underset{\sim}{x}}{d\tau}$ for a massive particle).

Also, for a photon, $\underset{\sim}{u} \cdot \underset{\sim}{u} = 0$ (not -1 as it would be for a massive particle).

In our problem the photon is emitted in the radial direction, so

$$\underset{\sim}{u} = \left(\frac{dt}{d\lambda}, \frac{dr}{d\lambda}, 0, 0 \right) \quad \text{in} \quad (t\, r\, \theta\, \phi) \text{ coords.}$$

↑ photon

We'll find these 2 components just like we did for the infalling particle.

First, we use the conservation law, (this is 9.58)

$$\left(1 - \frac{2M}{r}\right) \frac{dt}{d\lambda} = e, \qquad \frac{dt}{d\lambda} = \frac{e}{1 - \frac{2M}{r}} .$$

same one and for same reason as for the massive, infalling particle but e has a different value now since it refers to the photon.

Next using $\underset{\sim}{u} \cdot \underset{\sim}{u} = 0$, we find

$$\frac{dr}{d\lambda} = e .$$

The radial eqn is simple because the photon moves in the radial direction only; the more general case where there is angular motion is (9.63).

Now what is the value of e? If an observer with world velocity $\underset{\sim}{u}_{obs}$ receives a photon of ~~world velocity~~ 4-momentum $\underset{\sim}{u}_{photon}$, then the energy of that photon as seen by that observer is

$$E = - \underset{\sim}{u}_{obs} \cdot \underset{\sim}{u}_{photon} .$$

If the observer is the infalling particle, then

$$\underset{\sim}{u}_{obs} \longrightarrow \underset{\sim}{u}_{part} = \left(\frac{1}{1-\frac{2M}{r}}, -\sqrt{\frac{2M}{r}}, 0, 0 \right)$$

$$\underset{\sim}{u}_{photon} = \left(\frac{e}{1-\frac{2M}{r}}, e, 0, 0 \right)$$

where now r is the r coordinate where the photon is emitted. But in this case $E = E_* = \hbar\omega_*$ for the energy of the photon as seen by the infalling particle, so we get

$$E_* = \hbar\omega_* = -\underset{\sim}{u}_{part} \cdot \underset{\sim}{u}_{photon}$$

$$= -\left[g_{tt} \left(\frac{dt}{d\tau}\right)_{part} \left(\frac{dt}{d\lambda}\right)_{phot} + g_{rr} \left(\frac{dr}{d\tau}\right)_{part} \left(\frac{dr}{d\tau}\right)_{phot} \right]$$

$$= \left(1-\frac{2M}{r}\right)\left(\frac{1}{1-\frac{2M}{r}}\right)\left(\frac{e}{1-\frac{2M}{r}}\right) - \left(\frac{1}{1-\frac{2M}{r}}\right)\left(-\sqrt{\frac{2M}{r}}\right)(e)$$

$$= \frac{e}{1-\frac{2M}{r}} + \frac{e}{1-\frac{2M}{r}}\sqrt{\frac{2M}{r}} = \frac{e}{1-\frac{2M}{r}}\left(1+\sqrt{\frac{2M}{r}}\right)$$

$$= \frac{e}{1-\sqrt{\frac{2M}{r}}}.$$

(Notice that $\left(1-\sqrt{\frac{2M}{r}}\right)\left(1+\sqrt{\frac{2M}{r}}\right) = \left(1-\frac{2M}{r}\right)$.)

So,

$$\boxed{E_* = \hbar\omega_* \quad \frac{e}{1-\sqrt{\frac{2M}{r}}}}$$

useful intermediate result.

If the observer is the one at ∞, then ($r = R \gg M$)

$$\underset{\sim}{u}_{obs} \longrightarrow (1, 0, 0, 0)$$

since space is Minkowski there. So

$$E_\infty = \hbar\omega_\infty = -\underset{\sim}{u}_{\left(\substack{obs\ at \\ \infty}\right)} \cdot \underset{\sim}{u}_{photon}$$

$$= \left(1 - \frac{2M}{R}\right)(1)\frac{e}{\left(1 - \frac{2M}{R}\right)} - \left(\frac{1}{1 - \frac{2M}{R}}\right)(0)(e) = e$$

Note that $\frac{2M}{R} \to 0$ here (large R). The simple result is

$$\boxed{E_\infty = \hbar\omega_\infty = e}$$

so this gives us the physical interpretation of e.

Taking ratio of last 2 boxed eqns gives

$$\boxed{\frac{\omega_\infty}{\omega_*} = 1 - \sqrt{\frac{2M}{r}}}$$

where r is the coordinate at which the photon was emitted.

So this is the answer, except the problem wants to know the ratio $\frac{\omega_\infty}{\omega_*}$ as a function of t_R (time received) not r (radius emitted). So we need to express r as a function of t_R.

As I read the problem 12.16, it sounds like it wants you to find the answer exactly, whereas the result (12.11) refers only to the last few moments before oblivion. ~~To~~ It's easier to get (12.11) as an approximation in the last few moments than it is to get the exact answer, so I'll do that first. The method is the same one used in my lecture in class.

The idea is to expand $r(\tau)$ about the proper time τ at which the infalling particle crosses the event horizon. Choose the origin of τ so that $\tau = 0$ when $r = 2M$. The differential equation is

$$\frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}} \qquad \text{(bottom p. 4)}$$

so

$$r(\tau) = r(0) + \frac{dr}{d\tau}(0)\,\tau + \ldots$$

$$= 2M + (-1)\,\tau + \ldots$$

or

$$\boxed{r = 2M - \tau + O(\tau^2)} \quad \text{when } \tau \text{ small.}$$

Notice that we are interested in the period just <u>before</u> the particle crosses the event horizon, so $\tau < 0$ (but small) here, ie, we're interested in the limit $\tau \to 0$ from below. (This eqn is still true when τ goes positive, but that's after the infalling particle crosses $r = 2M$, when it is no longer visible to observer at ∞.)

Now use
$$r = 2M - \tau ,$$
$$\frac{r}{2M} = 1 - \frac{\tau}{2M} ,$$
$$\frac{2M}{r} = 1 + \frac{\tau}{2M} ,$$
$$\sqrt{\frac{2M}{r}} = 1 + \frac{\tau}{4M} ,$$
$$\frac{\omega_\infty}{\omega_*} = 1 - \sqrt{\frac{2M}{r}} = -\frac{\tau}{4M} ,$$

all with corrections of order τ^2. The last eqn is valid when $\tau < 0$. It gives us $\frac{\omega_\infty}{\omega_*}$ as a fn of τ, still not what we want since we want this ratio as a fn. of t_R. But remember that

$$\frac{\omega_\infty}{\omega_*} = \frac{d\tau}{dt}$$

(where now $t = t_R$)! So we get

$$\frac{d\tau}{dt} = -\frac{\tau}{4M} ,$$
$$\frac{d\tau}{\tau} = -\frac{dt}{4M} ,$$
$$\ln|\tau| = -\frac{t}{4M} + \text{const.}$$
$$|\tau| = \text{const } e^{-t/4M} ,$$
$$\frac{\omega_\infty}{\omega_*} = -\frac{\tau}{4M} = \frac{|\tau|}{4M} = \text{const } e^{-t/4M}$$

which is the result (12.11).

You can retrace these steps, but without making an expansion about $\tau = 0$ (at $r = 2M$), instead solving

$$\frac{dr}{d\tau} = -\sqrt{\frac{2M}{r}}$$

exactly, to get $r(\tau)$, then plug that into

$$\frac{\omega_\infty}{\omega_*} = \frac{d\tau}{dt} = 1 - \sqrt{\frac{2M}{r}} = f(\tau) \quad \text{some fn of } \tau \text{ since } r(\tau) \text{ known,}$$

then integrating this to get $t(\tau)$, inverting to get $\tau(t)$, and finally expressing $\frac{\omega_\infty}{\omega_*}$ exactly as a fn of t. The algebra is messy and the results are hardly more illuminating than the small τ approximation I have used, so I won't do it.

The problem asks if the answers would be the same if the photon emitted by the infalling particle were not emitted in the purely radial direction. The answer is no, because $\underset{\sim}{u} = \frac{d\underset{\sim}{x}}{d\lambda}$ for the photon would now have θ and ϕ components, and the r component would no longer be so simple.