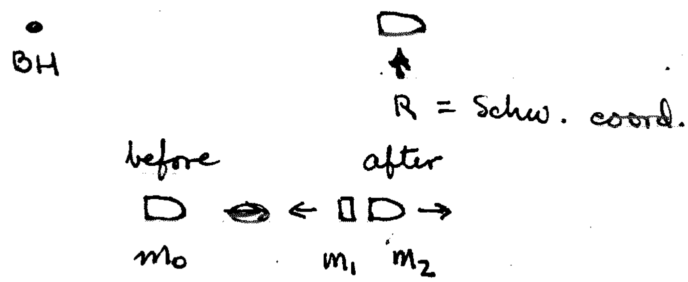
12.15. Hovering around black hole.

Then 

m_0, m_1, m_2 = rest masses. Piece m_2 leaves with velocity v_2 = esc. velocity. Actually v_2 is given by $\sqrt{\frac{2M}{R}}$, but we don't need to know that to do 90% of this problem. Let's just suppose v_2 is given (some number $0 < v_2 < 1$). The 90% of the problem is actually purely special relativity. We work in the local Lorentz frame of the (initially) stationary space ship, and refer quantities E, p, v to this frame.

We know v_2 but not m_2. Then

$$\gamma_2 = \frac{1}{\sqrt{1-v_2^2}} = \text{known.}$$

$$E_2 = m_2\gamma_2$$

$$p_2 = m_2\gamma_2 v_2 .$$

Again, rest mass is not conserved! So $m_1 + m_2 \neq m_0$!
But we do have conservation of energy and momentum. So,

$$E_1 + E_2 = m_0 \quad \Rightarrow \quad E_1 = m_0 - m_2\gamma_2$$

$$p_1 + p_2 = 0 \quad \Rightarrow \quad p_1 = -m_2\gamma_2 v_2 \qquad (p_1 < 0).$$

So, $$m_1 = \sqrt{E_1^2 - p_1^2} = \sqrt{(m_0 - m_2\gamma_2)^2 - m_2^2\gamma_2^2 v_2^2}$$

or

$$m_1 = \sqrt{m_0^2 - 2m_2\gamma_2 m_0 + \underbrace{m_2^2\gamma_2^2 - m_2^2\gamma_2^2 v_2^2}_{\hookrightarrow\ m_2^2\gamma_2^2(1-v_2^2) = m_2^2}}$$

so

$$m_1 = \sqrt{m_0^2 - 2m_0 m_2\gamma_2 + m_2^2}\,.$$

We know v_2, γ_2 and m_0 but not m_2, so we can't solve for m_1. But the problem doesn't ask us to solve for m_1 (or p_1 or E_1) it asks for the maximum value of $m_2 = f m_0$ (ie the max value of f) that can be achieved. But if we were given m_2 we could solve for m_1.

It's easy to show that if m_2 is increasing then m_1 is decreasing, and this is physically reasonable, too, since nonrelativistically we would have $m_1 + m_2 = m_0$. So the maximum value of m_2 must be the minimum value of m_1. What is that value? It certainly cannot be <0. Is 0 possible? To see, we set

$$m_2^2 - 2m_0\gamma_2 m_2 + m_0^2 = 0$$

and regard m_2 as unknown, so we get

$$m_2 = \gamma_2 m_0 \pm \sqrt{\underbrace{\gamma_2^2 m_0^2 - m_0^2}_{\hookrightarrow\ m_0^2(\gamma_2^2-1) = m_0^2 v_2^2\gamma_2^2}}.$$

$$m_2 = m_0(\gamma_2 \pm \gamma_2 v_2) = m_0\gamma_2(1 \pm v_2)$$

or,

$$m_2 = m_0\sqrt{\frac{1+v_2}{1-v_2}}, \quad m_0\sqrt{\frac{1-v_2}{1+v_2}}.$$

$\uparrow$ gives $m_2 > m_0$, impossible

$\uparrow$ gives $m_2 < m_0$ must be the solution.

So we get

$$m_2 = m_0\sqrt{\frac{1-v_2}{1+v_2}}$$

$$\Rightarrow \quad m_1 = 0$$

$$E_1 = m_0 - m_2\gamma_2 = m_0\left(1 - \frac{1}{\sqrt{1-v_2^2}}\sqrt{\frac{1-v_2}{1+v_2}}\right)$$

$$= m_0\left(1 - \frac{1}{1+v_2}\right) = m_0\frac{v_2}{1+v_2}.$$

$$p_1 = -m_2\gamma_2 v_2 = -m_0\sqrt{\frac{1-v_2}{1+v_2}}\frac{1}{\sqrt{1-v_2^2}}v_2$$

$$= -m_0\frac{v_2}{1+v_2}.$$

So $E_1 = |p_1|$ as should be when $m_1 = 0$. Fragment 1 is ejected at the speed of light.

Now to answer the problem,

$$f = \sqrt{\frac{1-v_2}{1+v_2}} = \sqrt{\frac{1-\sqrt{\frac{2M}{R}}}{1+\sqrt{\frac{2M}{R}}}}.$$

as $R \to 2M$ from above, $f \to 0$.