Physics 139: Problem Set 8 solutions

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Hartle 9.9

Find the relation between the rate of change of angular position of a particle in a circular orbit with respect to proper time and the Schwarzschild radius of the orbit. Compare with (9.46).

For a circular orbit of radius R, the proper angular velocity is

$$\frac{d\phi}{d\tau} = \frac{\ell}{R^2}.$$

We therefore only have to determine ℓ from the condition that a circular orbit lies at the minimum of the effective potential

$$\left.\frac{\partial V_{\text{eff}}}{\partial r}\right|_R = 0.$$

This gives

$$\ell^2 = \frac{MR^2}{R - 3M}$$

so that

$$\frac{d\phi}{d\tau} = \left(\frac{M}{R^3}\right)^{1/2} \left(1 - \frac{3M}{R}\right)^{-1/2}.$$

This is faster than $d\phi/dt$, given in (9.46). A clock on the circulating particle runs slow compared to a clock at infinity both because it is moving (time dilation) and because it is in a lower gravitational potential. The proper period is therefore *less* than the period in t. The proper speed must therefore be *greater*. Note that there can be no circular orbits with $R < 3M$.

Hartle 9.10

Find the linear velocity of a particle in a circular orbit of radius R in the Schwarzschild geometry that would be measured by a stationary observer stationed at one point on the orbit. What is its value at the ISCO?

There are several ways of doing this problem, each involving projecting the four-velocity of the particle onto the orthonormal basis of the observer. It's perhaps simplest to calculate the energy the observer measures and relate that to the velocity. From (7.53) and (5.44),

$$E = -\boldsymbol{p} \cdot \boldsymbol{u}_{\text{obs}} = \frac{m}{\sqrt{1 - V^2}}$$

where $\boldsymbol{p} = m\boldsymbol{u}$ is the particle's four-momentum, $\boldsymbol{u}_{\text{obs}}$ is the stationary observer's four-velocity, and we have assumed a rest mass m for the particle. $\boldsymbol{u}_{\text{obs}}$ can be found from the normalization condition, requiring that the observer is at rest:

$$\boldsymbol{u}_{\text{obs}} \cdot \boldsymbol{u}_{\text{obs}} = g_{tt} \left(u^t_{\text{obs}}\right)^2 = -1 \implies \boldsymbol{u}_{\text{obs}} = \left\{ \left(1 - \frac{2M}{R}\right)^{-1/2}, 0 \;, 0 \;, 0 \right\}$$

Plugging in, and using (9.47) and (9.48) for the particle's four-velocity, we find

$$E = -g_{tt} m u^t u^t_{\text{obs}} = m \left(1 - \frac{3M}{R}\right)^{1/2} \left(1 - \frac{2M}{R}\right)^{-1/2}.$$

Solving for the velocity, we have

$$V = \left(\frac{M}{R}\right)^{1/2} \left(1 - \frac{2M}{R}\right)^{-1/2}$$

At the innermost stable circular orbit (ISCO), $R = 6M$, so $V = 1/2$.

Hartle 9.11

A small perturbation of an unstable circular orbit will grow exponentially in time. Show that a displacement δr from the unstable maximum of the Schwarzschild effective potential will grow initially as

$$\delta r \propto e^{\tau/\tau_*}$$

where τ is the proper time along the particle's trajectory and τ_* is a constant. Evaluate τ_*. Explain its behavior as the radius of the orbit approaches $6M$.

In the neighborhood of its maximum at radius r_{max}, the effective potential behaves as

$$V_{\text{eff}}(r) = V_{\text{eff}}(r_{\text{max}}) + \frac{1}{2}\left(\frac{d^2 V_{\text{eff}}}{dr^2}\right)_{r_{\text{max}}} (\delta r)^2 + \;\cdots$$

where $\delta r \equiv r - r_{\text{max}}$. Denote $(d^2V_{texteft}/dr^2)_{r_{textmax}}$ by $-K^2$ since it is negative at the maximum of the potential. Eq. (9.29) becomes

$$\frac{1}{2}\left(\frac{d(\delta r)}{d\tau}\right)^2 - \frac{1}{2}K^2(\delta r)^2 = 0.$$

There are growing and decaying solutions to this upside-down harmonic oscillator. The growing solution behaves as

$$\delta r \propto e^{K\tau}$$

Thus the time constant τ_* is just $1/K$. Carrying out the derivatives explicitly, we find

$$\tau_* = \sqrt{\frac{r^5_{\text{max}}}{2\ell^2\left(r_{\text{max}} - 6M\right)}}$$

where, as before, r_{max} is the radius corresponding to the maximum of the effective potential. This diverges as r approaches $6M$ because the stable and unstable circular orbits are coalescing.

Hartle 9.12

A comet's orbit starts at infinity, goes around a relativistic star, then goes out to infinity again. The impact parameter at infinity is b. The radius of closest approach in Schwarzschild coordinates is R. What is the speed of the comet at closest approach as measured by a stationary observer at that point?

At the radius R of closest approach, the Schwarzschild coordinate components of the comet's four-velocity are ($\theta = \pi/2$):

$$\frac{dr}{d\tau} = 0, \ \frac{d\phi}{d\tau} = \frac{\ell}{R^2}$$

We can find the speed V from the energy measured by the stationary observer, which is

$$E = \frac{m}{\sqrt{1 - V^2}} = -\boldsymbol{p} \cdot \boldsymbol{u}_{\text{obs}}$$

where $\boldsymbol{p} = m\boldsymbol{u}$ is the four-momentum of the comet and $\boldsymbol{u}_{\text{obs}}$ is the four-velocity of the stationary observer

$$\boldsymbol{u}_{\text{obs}} = \left\{ \left(1 - \frac{2M}{R}\right)^{-1/2}, \ 0 \ , \ 0 \ , \ 0 \right\}$$

Thus

$$E = \left(1 - \frac{2M}{R}\right)^{1/2} m \ u^t$$

in which u^t can be calculated from the normalization condition and the above equations for $dr/d\tau$ and $d\phi/d\tau$, to give

$$u^t = \left(1 - \frac{2M}{R}\right)^{-1/2} \left(\frac{\ell^2}{R^2} + 1\right)^{1/2}$$

Combining these results, we have

$$V = \frac{\ell/R}{\sqrt{1 + (\ell/R)^2}}$$

Now we must express ℓ in terms of b and R. At large r, $\phi \approx b/r$, where b is the impact parameter,

$$\ell \equiv r^2 \frac{d\phi}{d\tau} = -b \left(\frac{dr}{d\tau}\right) = b\sqrt{e^2 - 1}$$

since $dr/d\tau$ is negative as the comet is heading towards the star. The last equality used the fact that at infinity, the metric approaches flat space, so that

$$\begin{aligned}
-1 &= -\left(1 - \frac{2M}{r}\right)\left(\frac{dt}{d\tau}\right)^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + r^2\left(\frac{d\phi}{d\tau}\right)^2 \\
&= -\left(1 - \frac{2M}{r}\right)^{-1} e^2 + \left(1 - \frac{2M}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2}{r^2} \\
&\to -e^2 + \left(\frac{dr}{d\tau}\right)^2 \\
\implies \frac{dr}{d\tau} &\to \pm\sqrt{e^2 - 1}
\end{aligned}$$

The energy per unit mass e is determined by applying the same normalization condition at the turning point:

$$\begin{aligned} -1 &= -\left(1-\frac{2M}{R}\right)^{-1} e^2 + \left(1-\frac{2M}{R}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 + \frac{\ell^2}{R^2} \\ &= -\left(1-\frac{2M}{R}\right)^{-1} e^2 + \frac{\ell^2}{R^2} \\ \implies e^2 &= \left(1-\frac{2M}{R}\right)\left(1+\frac{\ell^2}{R^2}\right). \end{aligned}$$

Solving for e with $\ell = b\sqrt{e^2-1}$ and plugging into the expression for V gives

$$V = b\,\sqrt{\frac{2M}{R}}\,\left(b^2 - R^2\right)^{-1/2}$$

Hartle 9.15

To find the first order in $1/c^2$ relativistic correction to the angle $\Delta\phi$ swept out by a planet in one bound orbit, factor $(1-2GM/c^2r)$ out of the denominator so that it can be written

$$\Delta\phi = 2\ell\int_{r_1}^{r_2}\frac{dr}{r^2}\left(1-\frac{2GM}{c^2r}\right)^{-1/2}\left[c^2e^2\left(1-\frac{2GM}{c^2r}\right)^{-1} - \left(c^2+\frac{\ell^2}{r^2}\right)\right]^{-1/2}$$

The factor in the brackets is the square root of a quantity quadratic in $1/r$ to order $1/c^2$. To derive the expression (9.55) evaluate this expression as follows:

a) Expand the factors of $(1-2GM/c^2r)$ in the above equation in powers of $1/c^2$ keeping only the leading order corrections to Newtonian quantities. Furthermore, expand e^2 to get

$$e^2 = 1 + \frac{2E_{\mathrm{Newt}}}{mc^2} + \cdots$$

b) Introduce the integration variable $u = 1/r$ and show that the integral can be put in the form

$$\Delta\phi = \left[1+2\left(\frac{GM}{c\ell}\right)^2\right] 2\int_{u_2}^{u_1}\frac{du}{\left[(u_1-u)(u-u_2)\right]^{1/2}}\left\{1+\frac{2GM}{c^2}u + \mathcal{O}\left(\frac{1}{c^4}\right)\right\}$$

c) The first term in the integral (including the 2) is just the one in (9.54) and equals 2π. Show that the second term gives $(\pi/2)(u_1+u_2)$ and that this equals $\pi GM/\ell^2$ to lowest order in $1/c^2$.

d) Combine these results to derive (9.55):

$$\delta\phi_{\mathrm{prec}} = 6\pi\left(\frac{GM}{c\ell}\right)^2 + \mathcal{O}\left(\frac{1}{c^4}\right)$$

The problem statement describes how to do the problem and the desired results.