Physics 139: Problem Set 12 solutions

May 2, 2014

Hartle 20.15

(Null Geodesics with Non-Affine Parametrization) As we showed in Section 8.3, when the tangent vector to a null geodesic $\mathbf{u}$ *is parameterized with an affine parameter* λ*, it obeys the geodesic equation*

$$\nabla_{\mathbf{u}}\mathbf{u} = 0$$

Show that even if a non-affine parameter is used

$$\nabla_{\mathbf{u}}\mathbf{u} = -\kappa\mathbf{u}$$

for some function κ *of the parameter* λ*.*

Let λ be the affine parameter so that $u^\alpha = dx^\alpha/d\lambda$ satisfies $\nabla_{\mathbf{u}}\mathbf{u} = 0$. If σ is a different, non-affine parameter so that $\lambda = \lambda(\sigma)$, then

$$u'^\alpha = \frac{dx^\alpha}{d\sigma} = \frac{dx^\alpha}{d\lambda}\frac{d\lambda}{d\sigma} \equiv f(\lambda)u^\alpha$$

Then

$$\begin{aligned}
\left(\nabla_{\mathbf{u}'}\mathbf{u}'\right)^\alpha &= u'^\beta\nabla_\beta u'^\alpha = fu^b\nabla_\beta\left(fu^\alpha\right) \\
&= f\left(u^\beta\nabla_\beta f\right)u^\alpha + f^2u^\beta\nabla_\beta u^\alpha \\
&= \frac{1}{2}\frac{df^2}{d\lambda}u^\alpha
\end{aligned}$$

in which we used the fact that $\nabla_{\mathbf{u}}\mathbf{u} = u^\beta\nabla_\beta\mathbf{u} = 0$ in the final equality. This shows the form of κ explicitly.

Hartle 20.16

(Surface Gravity of a Black Hole.) In the geometry of a spherical black hole, the Killing vector $\boldsymbol{\xi} = \partial/\partial t$ *corresponding to time translation invariance is tangent to the null geodesics that generate the horizon. If you have happened to work problem 15 you will know that this means*

$$\nabla_{\boldsymbol{\xi}}\boldsymbol{\xi} = -\kappa\boldsymbol{\xi}$$

for a constant of proportionality κ *which is called the surface gravity of the black hole. Evaluate this relation to find the value of* κ *for a Schwarzschild black hole in terms of its mass* M*. Be sure* *to use a coordinate system which is non-singular on the horizon such as the Eddington-Finkelstein coordinates discussed in Section 12.1.*

In Eddington-Finkelstein coordinates (v, r, θ, ϕ) we can take $\xi^\alpha = (1, 0, 0, 0)$. The partial derivative term in the covariant derivative clearly vanishes, so that

$$(\nabla_{\boldsymbol{\xi}} \boldsymbol{\xi})^\alpha = \xi^\beta \nabla_\beta \xi^\alpha = \nabla_v \xi^\alpha = -\Gamma^\alpha_{v\beta} \xi^\beta = -\Gamma^\alpha_{vv}$$

The Christoffel symbols can be evaluated from the derivatives of the metric. The inverse of the metric (keeping in mind that it's off-diagonal) has (v, r) components

$$g^{vv} = 0,\ g^{vr} = g^{rv} = 1,\ g^{rr} = \left(1 - \frac{2M}{r}\right)$$

The only non-zero ones are

$$\Gamma^v_{vv} = \frac{M}{r^2},\ \Gamma^r_{vv} = -\frac{2M}{r^2}\left(1 - \frac{2M}{r}\right)$$

On the horizon only Γ^v_{vv} is non-zero. This shows that $\nabla_{\boldsymbol{\xi}} \boldsymbol{\xi}$ is of the form $-\kappa \boldsymbol{\xi}$ with a constant κ that has the value

$$\kappa = \frac{1}{4M}$$

Hartle 20.18

(Killing's equation.) In Section 8.2 a Killing vector corresponding to a symmetry of a metric was defined in a coordinate system in which the metric was independent of one coordinate x^1. The components of the corresponding Killing vector $\boldsymbol{\xi}$ are then

$$\xi^\alpha = (0, 1, 0, 0)$$

By explicit calculation show that

$$\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0.$$

This is Killing's equation. It is a general characterization of Killing vectors in the sense that any solution corresponds to a symmetry of the metric.

The covariant derivative of the dual vector ξ_α is

$$\nabla_\beta \xi_\alpha = \frac{\partial \xi_\alpha}{\partial x^\beta} - \Gamma^\gamma_{\beta\alpha} \xi_\gamma$$

We also have

$$\xi_\alpha = g_{\alpha\beta} \xi^\beta = g_{\alpha\beta} \delta^\beta_A = g_{\alpha A}$$

Substituting this into the expression for the covariant derivative and plugging in the definition of the Christoffels, we have

$$\begin{aligned}
\nabla_\beta \xi_\alpha &= g_{\alpha A,\beta} - \Gamma^\gamma_{\beta\alpha} g_{\gamma A} \\
&= g_{\alpha A,\beta} - \frac{1}{2} g^{\gamma\mu} \left(g_{\beta\mu,\alpha} + g_{\alpha\mu,\beta} - g_{\alpha\beta,\mu}\right) g_{\gamma A} \\
&= g_{\alpha A,\beta} - \frac{1}{2} \delta^\mu_A \left(g_{\beta\mu,\alpha} + g_{\alpha\mu,\beta} - g_{\alpha\beta,\mu}\right) \\
&= g_{\alpha A,\beta} - \frac{1}{2} \left(g_{\beta A,\alpha} + g_{\alpha A,\beta} - g_{\alpha\beta,A}\right) \\
&= \frac{1}{2} \left(g_{\alpha A,\beta} - g_{\beta A,\alpha} + g_{\alpha\beta,A}\right)
\end{aligned}$$

in which we have used the comma notation for partial derivative, $T_{\alpha\beta\ldots\gamma,\mu} \equiv \partial T_{\alpha\beta\ldots\gamma}/\partial x^{\mu}$. Note that the last term in the above expression vanishes because the metric is independent of x^A. The remaining expression is antisymmetric under interchange of α and β. Thus

$$\nabla_\beta \xi_\alpha + \nabla_\alpha \xi_\beta = 0$$

as desired.

Littlejohn 4

If $\mathbf{V}$ *is a vector with components* V^μ*, then* $\nabla_{\mathbf{D}}\mathbf{V}$ *is another vector – the directional derivative of* $\mathbf{V}$ *along* $\mathbf{D}$*, with components* $(\nabla_{\mathbf{D}}\mathbf{V})^\mu = D^\alpha \nabla_\alpha V^\mu$*, where*

$$\nabla_\alpha V^\mu = \frac{\partial V^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\beta} V^\beta$$

Equivalently, we can say that $\nabla\mathbf{V}$ *is a tensor, the* covariant derivative *of* $\mathbf{V}$*, with components* $\nabla_\alpha V^\mu$*.*

Find an expression for $\nabla_\beta \nabla_\alpha V^\mu$*, the components of the second covariant derivative of* $\mathbf{V}$*. Express*

$$\nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu$$

in terms of the components of $\mathbf{V}$ *and the Riemann tensor,*

$$R^\mu{}_{\nu\alpha\beta} = \frac{\partial \Gamma^\mu_{\beta\nu}}{\partial x^\alpha} - \frac{\partial \Gamma^\mu_{\alpha\nu}}{\partial x^\beta} + \Gamma^\mu_{\alpha\sigma}\Gamma^\sigma_{\beta\nu} - \Gamma^\mu_{\beta\sigma}\Gamma^\sigma_{\alpha\nu}$$

The point is that the covariant derivatives along different directions do not commute, and the commutator involves the Riemann curvature tensor.

The second covariant derivative of $\mathbf{V}$ is the covariant derivative of $\nabla\mathbf{V}$:

$$\nabla_\beta \left(\nabla\mathbf{V}\right)_\alpha{}^\mu = \frac{\partial \left(\nabla\mathbf{V}\right)_\alpha{}^\mu}{\partial x^\beta} + \Gamma^\mu_{\beta\gamma}\left(\nabla\mathbf{V}\right)_\alpha{}^\gamma - \Gamma^\gamma_{\beta\alpha}\left(\nabla\mathbf{V}\right)_\gamma{}^\mu$$

Plugging in the components of the covariant derivative of $\mathbf{V}$ and using the comma notation for the partial derivatives, we have

$$\nabla_\beta \nabla_\alpha V^\mu = V^\mu_{,\alpha\beta} + \Gamma^\mu_{\alpha\gamma,\beta} V^\gamma + \Gamma^\mu_{\alpha\gamma} V^\gamma_{,\beta} + \Gamma^\mu_{\beta\gamma}\left(V^\gamma_{,\alpha} + \Gamma^\gamma_{\alpha\rho} V^\rho\right) - \Gamma^\gamma_{\beta\alpha}\left(V^\mu_{,\gamma} + \Gamma^\mu_{\gamma\rho} V^\rho\right)$$

Now we swap α and β and take the difference:

$$\begin{aligned}
\nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu &= \Gamma^\mu_{\beta\gamma,\alpha} V^\gamma - \Gamma^\mu_{\alpha\gamma,\beta} V^\gamma - \Gamma^\mu_{\alpha\gamma} V^\gamma_{,\beta} + \Gamma^\mu_{\beta\gamma} V^\gamma_{,\alpha} \\
&- \Gamma^\mu_{\beta\gamma}\left(V^\gamma_{,\alpha} + \Gamma^\gamma_{\alpha\rho} V^\rho\right) + \Gamma^\mu_{\alpha\gamma}\left(V^\gamma_{,\beta} + \Gamma^\gamma_{\beta\rho} V^\rho\right) \\
&+ \Gamma^\gamma_{\beta\alpha}\left(V^\mu_{,\gamma} + \Gamma^\mu_{\gamma\rho} V^\rho\right) - \Gamma^\gamma_{\alpha\beta}\left(V^\mu_{,\gamma} + \Gamma^\mu_{\gamma\rho} V^\rho\right)
\end{aligned}$$

The second partial derivatives of V^μ cancelled due to the commutativity of mixed partials. Furthermore, the last line will vanish due to the symmetry of the lower two indices of the Christoffel symbol, leaving

$$\begin{aligned}
\nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu &= \Gamma^\mu_{\beta\gamma,\alpha} V^\gamma - \Gamma^\mu_{\alpha\gamma,\beta} V^\gamma \\
&- \Gamma^\mu_{\alpha\gamma} V^\gamma_{,\beta} + \Gamma^\mu_{\beta\gamma} V^\gamma_{,\alpha} - \Gamma^\mu_{\beta\gamma} V^\gamma_{,\alpha} + \Gamma^\mu_{\alpha\gamma} V^\gamma_{,\beta} \\
&- \Gamma^\mu_{\beta\gamma}\Gamma^\gamma_{\alpha\rho} V^\rho + \Gamma^\mu_{\alpha\gamma}\Gamma^\gamma_{\beta\rho} V^\rho
\end{aligned}$$

in which we have expanded the terms in the parentheses and regrouped. Clearly, the second line will vanish, leaving

$$\begin{aligned}
\nabla_\alpha \nabla_\beta V^\mu - \nabla_\beta \nabla_\alpha V^\mu &= \Gamma^\mu_{\beta\gamma,\alpha} V^\gamma - \Gamma^\mu_{\alpha\gamma,\beta} V^\gamma - \Gamma^\mu_{\beta\gamma} \Gamma^\gamma_{\alpha\rho} V^\rho + \Gamma^\mu_{\alpha\gamma} \Gamma^\gamma_{\beta\rho} V^\rho \\
&= \left(\Gamma^\mu_{\beta\gamma,\alpha} - \Gamma^\mu_{\alpha\gamma,\beta} - \Gamma^\mu_{\beta\sigma} \Gamma^\sigma_{\alpha\gamma} + \Gamma^\mu_{\alpha\sigma} \Gamma^\sigma_{\beta\gamma} \right) V^\gamma \\
&= R^\mu{}_{\gamma\alpha\beta} V^\gamma
\end{aligned}$$

Hartle 21.2

(The Shape of the Tides) This problem concerns the shape of the tides raised by the moon in Newtonian gravity. Consider the freely falling frame following the center of mass of the Earth in its mutual orbit with the moon. (Neglect the slower motion of the Earth around the sun and the rotation of the Earth.) Assume the surface of the solid Earth is a sphere which is covered with a worldwide ocean.

a) *Explain why the surface of the ocean should be at an equal total gravitational potential.*

b) *Find a gravitational potential Φ_{tidal} that will reproduce the tidal gravitational force of the moon given in (c) of Box 21.1 and the gravitational force of the Earth on an ocean fluid element of mass m according to:*

$$\vec{F}_{\text{tidal}} = -m\nabla\Phi_{\text{tidal}}$$

c) *Find the difference $\delta h(\theta, \phi)$ between the depth of the ocean in the presence of the moon and in its absence caused by the tidal gravitational force of the moon. Use the usual polar angles with the z-axis pointing towards the moon. Express your answer in terms of the mass of the Earth, the mass of the moon, the distance between them, and the distance from the center of the Earth to the surface of the ocean were the moon not present.*

d) *Estimate the expected height of the ocean tides from your result in part c).*

e) *Answer the question at the end of Box 21.1.*

a)

Consider a cubical fluid element of mass m at the surface of the ocean. The gravitational force on it is $-m\nabla\Phi$. The pressure force results from the *difference* between the pressure on opposite sides. There is therefore no pressure force *along* surfaces of constant pressure. In fact, the pressure force is ∇p which is normal to surfaces of constant pressure. If the fluid element is not to accelerate, ∇p must lie along $m\nabla\Phi$ which means that surfaces of constant pressure are surfaces of constant potential. But the surface of the ocean is a constant pressure ($p = 0$) surface, so it must be an equipotential.

b)

Use (x, y, z) coordinates as the figure in Box 21.1. Let R be the radius of the Earth, d the distance from the Earth to the moon, and δh the difference between the height of the ocean and its mean height. Assume that $\delta h/R \ll 1$. The fluid element experiences forces from the gravitational fields of both the earth and the moon. The tidal acceleration of the fluid element relative to the center of the earth due to the moon (eqn b of Box 21.1) can be reproduced by the potential (per unit mass)

$$\Phi_{\text{tidal}} = \frac{1}{2}\left(\frac{\partial^2 \Phi_{\text{moon}}}{\partial x^i \partial x^j}\right)_0 x^i x^j = \frac{G M_{\text{moon}}}{2d^3}\left(x^2 + y^2 - 2z^2\right)$$

The earth's potential requires more care since it is singular at the origin of the coordinates. The gravitational force on a fluid element at (x, y, z) is

$$\vec{F}_{\text{Earth}} = -m\frac{G M_{\text{Earth}}}{R^3}\,(x, y, z)$$

For small variations in x, y, z near the surface, this can be reproduced by the potential (per unit mass)

$$\Phi_{\text{Earth}} = \frac{G M_{\text{Earth}}}{2R^3}\left(x^2 + y^2 + z^2\right)$$

The total force per mass on a fluid element is given by the combined potential $\Phi_{\text{Earth}} + \Phi_{\text{tidal}}$.

c)

The difference δh between the height of the ocean and its mean height $\tilde{h}$ is a function of θ alone due to axisymmetry. To first order in δh the total potential on a fluid element at $r = R + \delta h(\theta)$ is

$$\frac{G M_{\text{Earth}}}{2R^3} R^2 \left(1 + \frac{2\delta h(\theta)}{R}\right) + \frac{G M_{\text{moon}}}{2d^3} R^2 \left(1 - 3\cos^2\theta\right)$$

Along the surface of the ocean, this potential must be a constant, so that the θ dependence in the first term must cancel that of the second. A possible constant in δh is fixed by the requirement that it averages to zero. Thus

$$\frac{\delta h(\theta)}{R} = \frac{M_{\text{moon}}}{M_{\text{Earth}}}\left(\frac{R}{d}\right)^3 \left(\frac{3}{2}\cos^2\theta - 1\right)$$

This is positive in the direction of the moon and negative 90^0 away, as expected.

d)

The difference between the maximum and minimum heights is of order

$$\Delta h \sim \frac{3}{2} R \left(\frac{M_{\text{moon}}}{M_{\text{Earth}}}\right)\left(\frac{R}{d}\right)^3$$

For a ratio of the masses of the moon to the earth of $\approx 1/81$, $R \approx 6378$ km, and $d \approx 384,404$ km, this works out to be $\Delta h \sim 1$ m.

e)

As the Earth rotates under the tides, it loses rotational energy to friction. The rotation rate of the Earth is therefore slowing down. This requires that a torque be exerted on the tidally distorted earth by the moon. A little geometry convinces one that the tidal bulge must not point exactly at the moon but so that the high point of the tide *lags* the time the moon is at the zenith.