Physics 139: Problem Set 11 solutions

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Hartle 20.3

Use the transformation (7.2) connecting rectangular coordinates (t, x, y, z) for flat space to polar coordinates $(t, r, \theta\phi)$, to find explicitly the transformation laws giving the components (a^t, a^x, a^y, a^z) of a vector $\mathbf{a}$ *in terms of the components $(a^t, a^r, a^\theta, a^\phi)$ and the components (a_t, a_x, a_y, a_z) in terms of $(a_t, a_r, a_\theta, a_\phi)$.*

Since t is the same in both cases, a^t doesn't change, and neither does a_t. Writing out the transformation rule gives, for example,

$$a^x = \frac{\partial x}{\partial t}a^t + \frac{\partial x}{\partial r}a^r + \frac{\partial x}{\partial \theta}a^\theta + \frac{\partial x}{\partial \phi}a^\phi.$$

Then using $x = r\sin\theta\cos\phi$ gives

$$a^x = \sin\theta\cos\phi\ a^r + r\cos\theta\cos\phi\ a^\theta - r\sin\theta\sin\phi\ a^\phi$$

The results for a^y and a^z are obtained similarly:

$$\begin{aligned} a^y &= \sin\theta\sin\phi\ a^r + r\cos\theta\sin\phi\ a^\theta + r\sin\theta\cos\phi\ a^\phi \\ a^z &= \cos\theta\ a^r - r\sin\theta\ a^\theta \end{aligned}$$

The transformations for the inverse metric are obtained most efficiently just by lowering indices

$$a_x = a^x,\ a_y = a^y,\ a_z = a^z$$

and

$$a_r = a^r,\ a_\theta = r^2 a^\theta,\ a_\phi = r^2\sin^2\theta\ a^\phi$$

so that

$$\begin{aligned} a_x &= \sin\theta\cos\phi\ a_r + r^{-1}\cos\theta\cos\phi\ a_\theta - r^{-1}\sin^{-1}\theta\sin\phi\ a_\phi \\ a_y &= \sin\theta\sin\phi\ a_r + r^{-1}\cos\theta\sin\phi\ a_\theta + r^{-1}\sin^{-1}\theta\cos\phi\ a_\phi \\ a_z &= \cos\theta\ a_r - r^{-1}\sin\theta\ a_\theta \end{aligned}$$

Hartle 20.4

In the Schwarzschild geometry consider the following function:

$$f(x) = (5t^2 - 2r^2)/(2M)^2$$

where r and t are the usual Schwarzschild coordinates in which the metric has the form (9.9). Find the coordinate basis components $(\nabla f)^\alpha$ of the gradient of f.

The downstairs components of the gradient are

$$\nabla_\alpha f = \frac{\partial f}{\partial x^\alpha} = \{10t, -4r, 0, 0\}/(2M)^2$$

The upstairs components are

$$\begin{aligned}\nabla^\alpha f &= g^{\alpha\beta}\nabla_\beta f \\ &= \frac{1}{(2M)^2}\left\{-\left(1-\frac{2M}{r}\right)^{-1} 10t, -\left(1-\frac{2M}{r}\right)(4r), 0, 0\right\}\end{aligned}$$

Hartle 20.5

Eq. (20.81) gives the upstairs coordinate basis components of a set of four vectors $\{\mathbf{e}_{\hat{\alpha}}\}$ constituting an orthonormal frame in the Schwarzschild geometry.

a) *Verify explicitly that this is an orthonormal set of vectors.*

b) *Find the downstairs coordinate basis components of each of these vectors.*

c) *Find the upstairs coordinate basis components of the basis $\mathbf{e}^\alpha$ that is dual to the given set of basis vectors.*

d) *Consider a vector $\mathbf{a}$ with upstairs coordinate basis components*

$$a^\alpha = \{4, 3, 0, 0\}$$

at the point $\{0, 3M, 0, 0\}$. Find the components $a^{\hat{\alpha}}$ and $a_{\hat{\alpha}}$ of this vector in the given orthonormal frame.

a)

The scalar products can be computed from the given coordinate basis components as

$$\mathbf{e}_{\hat{\alpha}} \cdot \mathbf{e}_{\hat{\beta}} = g_{\alpha\beta}\,(\mathbf{e}_{\hat{\alpha}})^\alpha\,(\mathbf{e}_{\hat{\beta}})^\beta$$

and should equal $\eta_{\hat{\alpha}\hat{\beta}}$. For example

$$\begin{aligned}\mathbf{e}_{\hat{\tau}} \cdot \mathbf{e}_{\hat{r}} &= \left(1-\frac{2M}{r}\right)\left(1-\frac{2M}{r}\right)^{-1}\left(\frac{2M}{r}\right)^{1/2}\left(1-\frac{2M}{r}\right)^{-1} \\ &+ \left(1-\frac{2M}{r}\right)^{-1}\left[-\left(\frac{2M}{r}\right)^{1/2}\right]\cdot 1 = 0\end{aligned}$$

b)

The downstairs components are

$$(\mathbf{e}_{\hat{\alpha}})_\alpha = g_{\alpha\beta}(\mathbf{e}_{\hat{\alpha}})^\beta$$

The result is

$$\begin{aligned}
(\mathbf{e}_{\hat{\tau}})_\alpha &= \left\{-1, -\left(\frac{2M}{r}\right)^{1/2}\left(1-\frac{2M}{r}\right)^{-1}, 0, 0\right\} \\
(\mathbf{e}_{\hat{r}})_\alpha &= \left\{\left(\frac{2M}{r}\right)^{1/2}, \left(1-\frac{2M}{r}\right)^{-1}, 0, 0\right\} \\
(\mathbf{e}_{\hat{\theta}})_\alpha &= \{0, 0, r, 0\} \\
(\mathbf{e}_{\hat{\phi}})_\alpha &= \{0, 0, 0, r\sin\theta\}
\end{aligned}$$

c)

Since this is an orthonormal basis, the dual vectors are given by

$$e^{\hat{0}} = -e_{\hat{0}}, \ e^{\hat{i}} = e_{\hat{i}}$$

d)

The components $a_{\hat{\alpha}} = \mathbf{e}_{\hat{\alpha}} \cdot \mathbf{a}$ are

$$a_{\hat{\alpha}} = \left\{-3\left(\frac{4}{3}+3\left(\frac{2}{3}\right)^{1/2}\right), 3\left(\frac{4}{3}\left(\frac{2}{3}\right)^{1/2}+3\right), 0, 0\right\}$$

Then $a^{\hat{\alpha}}$ is given by

$$a^{\hat{\alpha}} = \left\{3\left(\frac{4}{3}+3\left(\frac{2}{3}\right)^{1/2}\right), 3\left(\frac{4}{3}\left(\frac{2}{3}\right)^{1/2}+3\right), 0, 0\right\}$$

Hartle 20.9

Show that the operation of contraction as exemplified by (20.40) is basis independent by showing that if carried out in another system of coordinates $x'^\alpha - x'^\alpha(x^\beta)$ the components of w^α transform correctly as a consequence of the transformation law for tensors.

The transformation rule for a tensor $t_{\alpha\beta}{}^{\gamma}$ is

$$t'_{\alpha\beta}{}^{\gamma} = \frac{\partial x^\delta}{\partial x'^\alpha}\frac{\partial x^\epsilon}{\partial x'^\beta}\frac{\partial x'^\gamma}{\partial x^\zeta} t_{\delta\epsilon}{}^{\zeta}$$

Contracting β and γ gives

$$\begin{aligned}
w'_\alpha \equiv t'_{\alpha\beta}{}^{\beta} &= \frac{\partial x^\delta}{\partial x'^\alpha}\frac{\partial x^\epsilon}{\partial x'^\beta}\frac{\partial x'^\beta}{\partial x^\zeta} t_{\delta\epsilon}{}^{\zeta} \\
&= \frac{\partial x^\delta}{\partial x'^\alpha}\frac{\partial x^\epsilon}{\partial x^\zeta} t_{\delta\epsilon}{}^{\zeta} \\
&= \frac{\partial x^\delta}{\partial x'^\alpha}\delta^\epsilon_\zeta t_{\delta\epsilon}{}^{\zeta} \\
&= \frac{\partial x^\delta}{\partial x'^\alpha} t_{\delta\epsilon}{}^{\epsilon} \\
&= \frac{\partial x^\delta}{\partial x'^\alpha} w_\delta
\end{aligned}$$

which is the correct transformation law for a vector. Thus, the contraction is basis independent.