Transforming the Christoffel Symbols.

Let x^μ and x'^μ be two coordinate systems. The geodesic equations in the two systems are

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta} \frac{dx^\alpha}{d\tau} \frac{dx^\beta}{d\tau} \tag{1}$$

and

$$\frac{d^2 x'^\mu}{d\tau^2} = -\Gamma'^\mu_{\alpha\beta} \frac{dx'^\alpha}{d\tau} \frac{dx'^\beta}{d\tau}. \tag{2}$$

We wish to find a relationship between $\Gamma^\mu_{\alpha\beta}$ and $\Gamma'^\mu_{\alpha\beta}$. The procedure is just the chain rule plus algebra.

We begin with

$$\frac{dx'^\mu}{d\tau} = \frac{\partial x'^\mu}{\partial x^\sigma} \frac{dx^\sigma}{d\tau}, \tag{3}$$

so that

$$\frac{d^2 x'^\mu}{d\tau^2} = \frac{d}{d\tau}\left(\frac{\partial x'^\mu}{\partial x^\sigma} \frac{dx^\sigma}{d\tau} \right)$$

$$= \frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\gamma} \frac{dx^\sigma}{d\tau} \frac{dx^\gamma}{d\tau} + \frac{\partial x'^\mu}{\partial x^\sigma} \frac{d^2 x^\sigma}{d\tau^2}$$

$$= \left(\frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\gamma} - \frac{\partial x'^\mu}{\partial x^\sigma} \Gamma^\sigma_{\lambda\gamma} \right) \frac{dx^\lambda}{d\tau} \frac{dx^\gamma}{d\tau}, \tag{4}$$

where we have used (1) and juggled indices slightly.

Now using the chain rule,

$$\frac{dx^\lambda}{d\tau} = \frac{\partial x^\lambda}{\partial x'^\alpha}\frac{dx'^\alpha}{d\tau} \; ; \qquad \frac{dx^\gamma}{d\tau} = \frac{\partial x^\gamma}{\partial x'^\beta}\frac{dx'^\beta}{d\tau} , \qquad (5)$$

and substituting (2), we find

$$\frac{d^2 x'^\mu}{d\tau^2} = \left(\frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\gamma} - \frac{\partial x'^\mu}{\partial x^\sigma} \Gamma^\sigma_{\lambda\gamma} \right) \frac{\partial x^\lambda}{\partial x'^\alpha} \frac{\partial x^\gamma}{\partial x'^\beta} \frac{dx'^\alpha}{d\tau} \frac{dx'^\beta}{d\tau} , \qquad (6)$$

or

$$\boxed{\Gamma'^\mu_{\alpha\beta} = \left(\frac{\partial x'^\mu}{\partial x^\sigma} \Gamma^\sigma_{\lambda\gamma} - \frac{\partial^2 x'^\mu}{\partial x^\lambda \partial x^\gamma} \right) \frac{\partial x^\lambda}{\partial x'^\alpha} \frac{\partial x^\gamma}{\partial x'^\beta}} \qquad (7)$$

This is the desired transformation law for Γ. Notice that the first term is the transformation law for a tensor with one upper and two lower indices; but the presence of the second term, which involves the derivatives of the Jacobian matrices, shows that Γ does not transform as a tensor. Thus, it is possible for Γ to vanish in one coordinate system but not another.

The inverse transformation is obtained simply by swapping x and x', and Γ and Γ'. Thus,

$$\boxed{\Gamma^\mu_{\alpha\beta} = \left(\frac{\partial x^\mu}{\partial x'^\sigma} \Gamma'^\sigma_{\lambda\gamma} - \frac{\partial^2 x^\mu}{\partial x'^\lambda \partial x'^\gamma} \right) \frac{\partial x'^\lambda}{\partial x^\alpha} \frac{\partial x'^\gamma}{\partial x^\beta}} \qquad (8)$$