Local Inertial Frames and Riemann Normal Coordinates

What Hartle calls a "Local Inertial Frame" (LIF) is actually a coordinate system in the neighborhood of a point with special properties. See p. 140. The construction of these coordinates does involve a special frame at one point, but the coordinate system covers a finite region about that point. Therefore I prefer to refer to the coordinates as "Riemann Normal Coordinates" (RNC for short), which is their proper name, rather than L.I.F.

The idea of RNC is that since a small region of space-time around a point (an event) P is approximately flat, it should be possible to introduce special coordinates in a neighborhood of P that look as close as possible to the usual $(txyz)$ coordinates of special relativity. In these notes we'll write $\bar{x}^\mu$ for these coordinates (the RNC), while letting x^μ (without the overbar) represent an arbitrary system of coordinates. In particular, it will turn out that if we Taylor expand the metric $\bar{g}_{\mu\nu}(\bar{x})$ in RNC about the point P, we get

$$\left.\begin{aligned} &\bar{g}_{\mu\nu}(P) = \eta_{\mu\nu} \\ &\frac{\partial \bar{g}_{\mu\nu}}{\partial \bar{x}^\alpha}(P) = 0, \end{aligned}\right\} \qquad (1)$$

so that

$$\bar{g}_{\mu\nu}(x) = \eta_{\mu\nu} + 0 + \Theta(\bar{x}^\mu - \bar{x}^\mu_P)^2. \qquad (2)$$

In fact, we'll arrange things so that $\bar{x}^\mu_P$ (the RNC of point P itself) is 0, so the last term is just $\Theta(\bar{x}^\mu)^2$. The construction of RNC depends on the choice of P; different P's give different RNC's.

Let $\underset{\sim}{e}_\mu(P) = \left.\frac{\partial}{\partial x^\mu}\right|_P$ be the coordinate basis vectors (in coordinates x^μ), evaluated at P. These satisfy

$$\underset{\sim}{e}_\mu(P) \cdot \underset{\sim}{e}_\nu(P) = g_{\mu\nu}(P), \qquad (3)$$

where $g_{\mu\nu}$ is the metric tensor in the x^μ coordinates. By forming linear combinations of these basis vectors we can get an orthonormal frame at P, call it $\underset{\sim}{e}_{\hat{\mu}}(P)$. Let the linear combinations be specified by a matrix $E_{\hat{\mu}}{}^{\nu}$,

$$\underset{\sim}{e}_{\hat{\mu}}(P) = E_{\hat{\mu}}{}^{\nu}\, \underset{\sim}{e}_\nu(P) \qquad (4).$$

The matrix $E_{\hat{\mu}}{}^{\nu}$ is not a function of x, it only operates at the one point P. To determine $E_{\hat{\mu}}{}^{\nu}$ we require

$$\underset{\sim}{e}_{\hat{\mu}}(P) \cdot \underset{\sim}{e}_{\hat{\nu}}(P) = \eta_{\mu\nu} = E_{\hat{\mu}}{}^{\alpha}\, g_{\alpha\beta}(P)\, E_{\hat{\nu}}{}^{\beta}. \qquad (5)$$

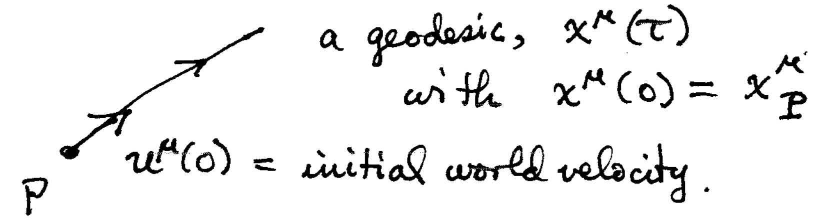
In matrix language, this is

$$\eta = E\, g\, E^T. \qquad (6)$$

To find E we first diagonalize the symmetric matrix $g_{\mu\nu}(P)$ with an orthogonal matrix, and then multiply this by a scaling transformation (a diagonal matrix with positive diagonal elements) that reduce the eigenvalues to ± 1. The result is $\eta = \text{diag}(-1, +1, +1, +1)$.

So $E_{\hat{\mu}}{}^{\nu}$ becomes known; it is a matrix of numbers.

Next, we solve the geodesic equations for geodesics that start at P, with initial world velocity $u^\mu(0)$. We assume $\tau = 0$ at P, so $u^\mu(0)$ means $u^\mu(P)$. We do this in the original coordinates x^μ.



For simplicity we'll just talk about time-like geodesics, but we can also construct space-like and light-like geodesics out of P as well.

It's easy to expand the solution $x^\mu(\tau)$ out to second order in τ. Since the equation of motion is

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}(x)\,\frac{dx^\alpha}{d\tau}\,\frac{dx^\beta}{d\tau}, \qquad (7)$$

we have

$$x^\mu(0) = x^\mu_P, \qquad \frac{dx^\mu}{d\tau}(0) = u^\mu(0), \qquad (8a)$$

and

$$\frac{d^2x^\mu}{d\tau^2}(0) = -\Gamma^\mu_{\alpha\beta}(0)\, u^\alpha(0)\, u^\beta(0). \qquad (8b).$$

So, the Taylor series of the geodesic is

$$x^\mu(\tau) = x^\mu(P) + \tau u^\mu(0) - \frac{\tau^2}{2}\Gamma^\mu_{\alpha\beta}(0)\, u^\alpha(0)\, u^\beta(0) + O(\tau^3) \qquad (9)$$

Now let's find the components of the initial world velocity in terms of the ON frame $\underset{\sim}{e}_{\hat{\mu}}(P)$. The relation is

$$\underset{\sim}{u}(0) = u^{\hat{\mu}}(0)\underset{\sim}{e}_{\hat{\mu}}(P) = u^\nu(0)\, \underset{\sim}{e}_{\hat{\nu}}(P)$$

$$= u^{\hat{\mu}}(0)\, E_{\hat{\mu}}{}^{\nu}\, \underset{\sim}{e}_\nu(P), \qquad (10)$$

or,

$$u^\nu(0) = E_{\hat{\mu}}{}^{\nu}\, u^{\hat{\mu}}(0). \qquad (11)$$

Plugging this into (9) gives

$$x^\mu(\tau) = x^\mu(P) + \tau E_{\hat{\alpha}}{}^{\mu}\, u^{\hat{\alpha}}(0) - \frac{\tau^2}{2}\Gamma^\mu_{\gamma\delta}\, E_{\hat{\alpha}}{}^{\gamma}\, E_{\hat{\beta}}{}^{\delta}\, u^{\hat{\alpha}}(0)\, u^{\hat{\beta}}(0) + \ldots \qquad (12)$$

where we've juggled indices somewhat.

Now we define the RNC by

$$\bar{x}^{\alpha} = u^{\hat{\alpha}}(0)\tau, \qquad (13)$$

so that

$$x^{\mu} = x^{\mu}_{P} + E_{\hat{\alpha}}{}^{\mu}\,\bar{x}^{\alpha} - \frac{1}{2}\Gamma^{\mu}_{\gamma\delta}(0)E_{\hat{\alpha}}{}^{\gamma}E_{\hat{\beta}}{}^{\delta}\,\bar{x}^{\alpha}\bar{x}^{\beta}$$

$$+\, O(\bar{x}^3) \qquad (14).$$

This is an explicit formula connecting the original coordinates x^{μ} and the RNC $\bar{x}^{\mu}$, carried to 2nd order in the RNC.

It is an exercise to show that under this coordinate transformation, the metric $\bar{g}_{\mu\nu}(\bar{x})$ in RNC has the form shown in (2).

Equation (13) is exact (not a Taylor series) because it is the definition of the RNC. It means that to find the RNC of a point with coordinates x^{μ}, call it Q, you first find the geodesic connecting P to Q, you then take the initial world velocity of this geodesic $u^{\mu}(0)$, then you convert it to the ON basis using (11) to get $u^{\hat{\mu}}(0)$, then $\bar{x}^{\mu}$ is defined by (13), where τ is the amount of proper time between P and Q. The Taylor series (14) expresses this process through $O(\tau^2)$.

This means that the equation of a geodesic starting at P in RNC is

$$\frac{d^2\bar{x}^\alpha}{d\tau^2} = 0 . \qquad (15)$$

But this must equal $-\bar{\Gamma}^\alpha_{\mu\nu}(\bar{x})\,\frac{d\bar{x}^\mu}{d\tau}\frac{d\bar{x}^\nu}{d\tau}$; so along the geodesic, where

$$\frac{d\bar{x}^\mu}{d\tau} = u^{\hat{\mu}}(0) = \text{const}, \qquad (16)$$

we have

$$\bar{\Gamma}^\alpha_{\mu\nu}\left(\bar{x}(\tau)\right) u^{\hat{\mu}}(0)\, u^{\hat{\nu}}(0) = 0 . \qquad (17)$$

This does not imply that $\bar{\Gamma}^\alpha_{\mu\nu}(\bar{x}(\tau)) = 0$, only that one component vanishes. However, at the special point $\bar{x} = 0$, that is, at P itself, $\bar{\Gamma}^\alpha_{\mu\nu} = 0$, because (17) applies there for all choices of initial $u^{\hat{\mu}}(0)$. Thus we obtain

$$\bar{\Gamma}^\alpha_{\mu\nu}(0) = 0 . \qquad (18).$$

Now, $\Gamma^\alpha_{\mu\nu}$ (in any coordinates) are proportional to the derivatives of $g_{\mu\nu}$:

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\beta}\left(\frac{\partial g_{\beta\mu}}{\partial x^\nu} + \frac{\partial g_{\beta\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\beta}\right), \qquad (19).$$

so if the derivatives of $g_{\mu\nu}$ vanish at a point, then so does $\Gamma^\alpha_{\mu\nu}$. The converse is also true; if $\Gamma^\alpha_{\mu\nu}$ vanishes at a point, then so does $\frac{\partial g_{\mu\nu}}{\partial x^\alpha}$.

Therefore, in RNC at point P $(\bar{x}=0)$, we have

$$\frac{\partial \bar{g}_{\mu\nu}}{\partial \bar{x}^\alpha}(P) = 0.$$

This proves the second part of (1).