Here is the relation between the directional or covariant derivative ∇_D along D and the covariant derivative $D/D\sigma$ along a curve $x^\mu(\sigma)$. Suppose V is a vector field and a curve $x^\mu = x^\mu(\sigma)$ is given. Then V is defined along the curve, and

$$\frac{DV^\mu}{D\sigma} = \frac{dV^\mu}{d\sigma} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\sigma}V^\beta. \tag{26}$$

On the other hand,

$$\nabla_D V^\mu = D^\alpha \frac{\partial V^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\beta} D_\alpha V^\beta. \tag{27}$$

Now if $D^\alpha = dx^\alpha/d\sigma$ is the tangent vector along the curve, then the first term in (27) becomes

$$\frac{dx^\alpha}{d\sigma}\frac{\partial V^\mu}{\partial x^\alpha} = \frac{dV^\mu}{d\sigma}, \tag{28}$$

the first term of (26). The second terms are equal, too, so

$$\frac{DV^\mu}{D\sigma} = \nabla_D V^\mu \qquad \text{when } D^\alpha = \frac{dx^\alpha}{d\sigma}. \tag{29}$$

On the other hand, if V^μ is only defined along the curve $x^\mu(\sigma)$, then the partial derivatives $\frac{\partial V^\mu}{\partial x^\alpha}$ are meaningless, so $\nabla_D V^\mu$ is not defined. Hartle uses the notation $\nabla_D V$ even when V is only defined along a curve. For example, he writes $\nabla_u u$ when $u^\mu = \frac{dx^\mu}{d\tau}$ is the unit tangent vector to a world line. What he really means is $\frac{Du}{D\tau}$. Thus, he writes the geodesic equation as $\nabla_u u = 0$, whereas I would write $\frac{Du}{D\tau} = 0$.

Also, he writes the equation geodesic deviation as

$$\nabla_u \nabla_u \xi^\mu = - R^\mu{}_{\cdot\, \nu\alpha\beta}\, u^\nu \xi^\alpha u^\beta, \qquad (30)$$

whereas I would write

$$\frac{D^2 \xi^\mu}{D\tau^2} = - R^\mu{}_{\cdot\, \nu\alpha\beta}\, u^\nu \xi^\alpha u^\beta. \qquad (31)$$

Here is some more notation concerning the covariant derivative. Let $\{e_\mu\}$ be a collection of basis vectors. Often this is a coordinate basis, but the same ~~logic~~ notation applies in orthonormal bases. Then we write

$$\nabla_{e_\mu} \equiv \nabla_\mu . \qquad (32)$$

For example,

$$\nabla_\alpha V^\mu \equiv \nabla_{e_\alpha} V^{\cancel{\beta}\mu} = \frac{\partial V^{\cancel{\beta}\mu}}{\partial x^\alpha} + \Gamma^\mu_{\alpha\beta} V^\beta. \qquad (33).$$

∇_α is the directional covariant derivative in the direction given by the basis vector e_α.

The notation $\nabla_D V^\alpha$ or $\nabla_\beta V^\alpha$ or $\frac{DV^\alpha}{D\tau}$ is a little tricky, because it looks like a operator, ∇_D or ∇_β or $\frac{D}{D\tau}$, is being applied to the components V^α of vector V. Actually, the operator is applied to the vector V, and then the α-th component is taken to get $\nabla_D V^\alpha$, $\nabla_\beta V^\alpha$, or $\frac{DV^\alpha}{D\tau}$. That is, to be careful about the meaning,

$$\nabla_D V^\alpha = (\nabla_D V)^\alpha, \qquad \nabla_\beta V^\alpha = (\nabla_\beta V)^\alpha,$$

$$\frac{DV^\alpha}{D\tau} = \left(\frac{DV}{D\tau}\right)^\alpha. \tag{34}$$

On the other hand, when we write $\frac{dV^\alpha}{d\tau}$, we really mean the τ-derivative of the component V^α along a curve $x(\tau)$, that is, we take the component first and then apply $\frac{d}{d\tau}$. (In fact, $\frac{dV}{d\tau}$ has no meaning.)

We have defined the action of $\frac{D}{D\sigma}$ and ∇_D on vectors. The action can be extended to other types of tensors by using two rules:

① If f is a scalar, then

$$\nabla_D f = D^\alpha \frac{\partial f}{\partial x^\alpha} = Df \tag{35}$$

$$\frac{Df}{D\tau} = \frac{df}{d\tau} \tag{36}$$

② ∇_D and $\frac{D}{D\sigma}$ obey the Leibnitz or chain rule. (37)

For example, let ω be a dual vector with components ω_μ, and V a vector with components V^μ. Then

$$\langle \omega, V \rangle = \omega(V) = \omega_\mu V^\mu = \text{a scalar}, \tag{38}$$

so

$$\nabla_D(\omega_\mu V^\mu) = D^\alpha \frac{\partial(\omega_\mu V^\mu)}{\partial x^\alpha}$$

$$= D^\alpha\left(\frac{\partial \omega_\mu}{\partial x^\alpha} V^\mu + \omega_\mu \frac{\partial V^\mu}{\partial x^\alpha}\right). \tag{39}$$

But by the chain rule we have

$$\nabla_D (\omega_\mu V^\mu) = (\nabla_D \omega_\mu) V^\mu + \omega_\mu (\nabla_D V^\mu)$$

$$= (\nabla_D \omega_\mu) V^\mu + \omega_\mu D^\alpha \left(\frac{\partial V^\mu}{\partial x^\alpha} + \Gamma^\mu_{\alpha\beta} V^\beta \right) \qquad (40)$$

Comparing (39) and (40), the term $\omega_\mu D^\alpha \frac{\partial V^\mu}{\partial x^\alpha}$ cancels, and we have

$$D^\alpha \frac{\partial \omega_\mu}{\partial x^\alpha} V^\mu = (\nabla_D \omega_\mu) V^\mu + \underbrace{\omega_\mu D^\alpha \Gamma^\mu_{\alpha\beta} V^\beta}_{\omega_\beta D^\alpha \Gamma^\beta_{\alpha\mu} V^\mu} \qquad (41)$$

or, since V^μ is arbitrary,

$$\boxed{\nabla_D \omega_\mu = D^\alpha \left(\frac{\partial \omega_\mu}{\partial x^\alpha} - \Gamma^\beta_{\alpha\mu} \omega_\beta \right)} \qquad (42)$$

We have derived the formula for the covariant derivative of a dual vector. Similarly,

$$\frac{D\omega_\mu}{D\sigma} = \frac{d\omega_\mu}{D\sigma} - \Gamma^\beta_{\alpha\mu} \frac{dx^\alpha}{d\sigma} \omega_\beta . \qquad (43).$$

Using this method we can find the covariant derivative of any tensor with any number of indices. In general, we find the components of the covariant derivative as the ordinary derivative of the components, plus one correction term in Γ with a + sign for each upper index, plus one correction term in Γ with a − sign for each lower index. For example, with a third rank tensor $T^{\mu}{}_{\alpha\beta}$, we have

$$\nabla_\gamma T^{\mu}{}_{\cdot\alpha\beta} = \frac{\partial T^{\mu}{}_{\cdot\alpha\beta}}{\partial x^\gamma} + \Gamma^{\mu}_{\gamma\sigma} T^{\sigma}{}_{\cdot\alpha\beta} - \Gamma^{\sigma}_{\gamma\alpha} T^{\mu}{}_{\cdot\sigma\beta}$$

$$- \Gamma^{\sigma}_{\gamma\beta} T^{\mu}{}_{\cdot\alpha\sigma} . \qquad (44)$$

A scalar has no correction terms because it has no indices; see (35) and (36).

A special case is the metric tensor. Its covariant derivative is

$$\nabla_\mu g_{\alpha\beta} = \frac{\partial g_{\alpha\beta}}{\partial x^\mu} - \Gamma^{\sigma}_{\mu\alpha} g_{\sigma\beta} - \Gamma^{\sigma}_{\mu\beta} g_{\alpha\sigma} . \qquad (45)$$

But the two correction terms are

$$-\frac{1}{2} g^{\sigma\tau}\left(\frac{\partial g_{\tau\mu}}{\partial x^\alpha} + \frac{\partial g_{\tau\alpha}}{\partial x^\mu} - \frac{\partial g_{\mu\alpha}}{\partial x^\tau}\right) g_{\sigma\beta}$$

$$-\frac{1}{2} g^{\sigma\tau}\left(\frac{\partial g_{\tau\mu}}{\partial x^\beta} + \frac{\partial g_{\tau\beta}}{\partial x^\mu} - \frac{\partial g_{\mu\beta}}{\partial x^\tau}\right) g_{\alpha\sigma} . \qquad (46).$$

But $g^{\sigma\tau} g_{\sigma\beta} = \delta^\tau_\beta$ and $g^{\sigma\tau} g_{\alpha\sigma} = \delta^\tau_\alpha$, so this becomes

$$-\frac{1}{2}\left(\cancel{\frac{\partial g_{\beta\mu}}{\partial x^\alpha}} + \frac{\partial g_{\beta\alpha}}{\partial x^\mu} - \cancel{\frac{\partial g_{\mu\alpha}}{\partial x^\beta}} + \cancel{\frac{\partial g_{\alpha\mu}}{\partial x^\beta}} + \frac{\partial g_{\alpha\beta}}{\partial x^\mu} - \cancel{\frac{\partial g_{\mu\beta}}{\partial x^\alpha}}\right)$$

$$= -\frac{\partial g_{\beta\alpha}}{\partial x^\mu} , \qquad (47)$$

which cancels the first term in (45). Thus we conclude,

$$\boxed{\nabla_\mu g_{\alpha\beta} = 0} . \qquad (48)$$

The covariant derivative of the metric tensor vanishes because of the way the connection was defined, in terms of Riemann normal coordinates, which are based on geodesics. Similarly, we have

$$\frac{D g_{\mu\nu}}{D\sigma} = 0, \tag{49}$$

along some curve $x^\mu(\sigma)$.

Because of (48), the process of raising or lowering indices commutes with the covariant derivative. For example,

$$\begin{aligned} g_{\mu\nu} \nabla_\alpha V^\mu &= \nabla_\alpha \left(g_{\mu\nu} V^\mu \right) - \left(\nabla_\alpha g_{\mu\nu} \right) V^\nu \\ &= \nabla_\alpha V_\nu - 0 = \nabla_\alpha V_\nu , \end{aligned} \tag{50}$$

where we use the chain rule for ∇_α and (48). Similarly,

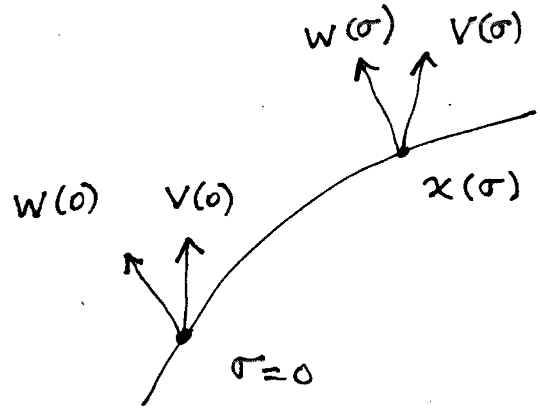
$$g^{\mu\nu} \nabla_\alpha \omega_\nu = \nabla_\alpha \omega^\mu . \tag{51}$$

We can also raise and lower the index on the ∇; for example

$$\nabla^\alpha \equiv g^{\alpha\beta} \nabla_\beta = \nabla_\beta g^{\alpha\beta} \tag{52}$$

when applied to any tensor.

An important consequence of (49) is the preservation of the scalar product under parallel transport along a curve. Let V^μ and W^μ be two vectors that are parallel transported along a curve $x^\mu(\sigma)$, with initial condition $V^\mu(0)$ and $W^\mu(0)$ at $\sigma=0$.



Then both V and W satisfy the equation of parallel transport:

$$\frac{DV^\mu}{D\sigma} = 0, \quad \frac{DW^\mu}{D\sigma} = 0. \tag{53}$$

Lowering the index on the W eqn, we have

$$\frac{DW_\mu}{D\sigma} = 0, \tag{54}$$

which implies

$$V^\mu \frac{DW_\mu}{D\sigma} + \frac{DV^\mu}{D\sigma} W_\mu = \frac{D}{D\sigma}\left(V^\mu W_\mu\right) = \frac{d}{d\sigma}\left(V^\mu W_\mu\right) = 0, \tag{55}$$

because $V^\mu W_\mu$ is a scalar. Thus

$$V^\mu(\sigma)\, W_\mu(\sigma) = V^\mu(0)\, W_\mu(0). \tag{56}$$

The scalar product is preserved by parallel transport.

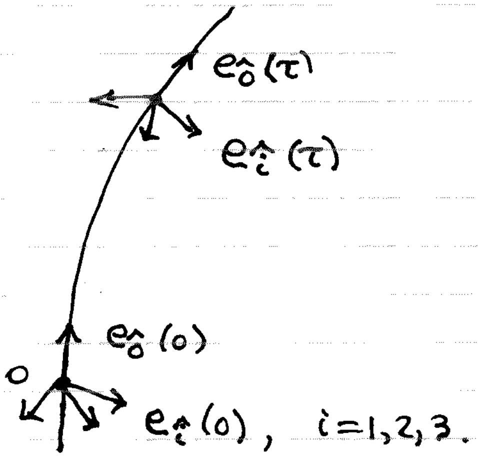
In particular, let the curve be a geodesic $x^\mu(\tau)$, and let $V^\mu = W^\mu = u^\mu = dx^\mu/d\tau$. Then

$$u^\mu(\tau)\, u_\mu(\tau) = u^\mu(0)\, u_\mu(0) = -1, \tag{57}$$

and the normalization of u is preserved by the geodesic parallel transport.

More generally, let $e_{\hat\mu}(0)$ be an orthonormal frame

at one point along a geodesic, with $e_{\hat{0}}(0) = u(0)$:
and let all 4 vectors be parallel transported along the curve, so that



$$\frac{D e_{\hat{\alpha}}}{D\tau} = 0, \qquad \alpha = 0,1,2,3. \qquad (58)$$

This means

$$\frac{D (e_{\hat{\alpha}})^{\mu}}{D\tau} = \frac{d}{d\tau}(e_{\hat{\alpha}})^{\mu} + \Gamma^{\mu}_{\beta\sigma} \frac{dx^{\beta}}{d\tau}(\hat{e}_{\hat{\alpha}})^{\sigma} = 0. \qquad (59).$$

Since we have

$$e_{\hat{\alpha}}(0) \cdot e_{\hat{\beta}}(0) = \eta_{\alpha\beta} \qquad (60)$$

(an orthonormal frame at $\tau=0$), we have

$$e_{\hat{\alpha}}(\tau) \cdot e_{\hat{\beta}}(\tau) = \eta_{\alpha\beta} \qquad (61)$$

(it remains an orthonormal frame all along the geodesic.) This frame is sometimes called the Fermi-transported frame; Hartle refers to it as a "freely falling frame." Physically, if $e_{\hat{i}}(0)$ for some $i = 1, 2$ or 3 is the spatial direction of a gyroscope attached to our freely falling particle, whose world line is the geodesic $x^{\mu}(\tau)$, then the gyroscope continues to point in the direction $e_{\hat{i}}(\tau)$ for later τ.

Here is an application of the covariant derivative. If a particle is not in free fall, so that $x^{\mu}(\tau)$ is not a geodesic, the $\frac{Du^\mu}{D\tau} \neq 0$. In fact, $\frac{Du^\mu}{D\tau}$ is interpreted as the 4-acceleration. This is logical, since if we used R.N.C., $\bar{x}^\mu$ which are as close as we can get to a Lorentz frame in general relativity, in which $\bar{\Gamma}^\mu_{\alpha\beta} = 0$, we would have

$$\frac{D\bar{u}^\mu}{D\tau} = \frac{d\bar{u}^\mu}{d\tau}. \tag{62}$$

We recall in special relativity that 4-acceleration $a^\mu = \frac{du^\mu}{d\tau}$ is a purely spacelike vector such that $a^\mu a_\mu = g^2$, where g is the magnitude of the acceleration as felt by an observer riding with particle. See Example 20.8 in the book, where Hartle calculates the proper acceleration of an observer hovering at radius R in the Schwarzschild geometry; the answer is

$$g = \frac{M}{R^2} \frac{1}{\sqrt{1-\frac{2M}{R}}}. \tag{63}$$

It approaches the Newtonian result as $R \to \infty$, but as $R \to 2M$ the acceleration experienced by the observer (and the work the rocket engines must do to hover) goes to ∞.