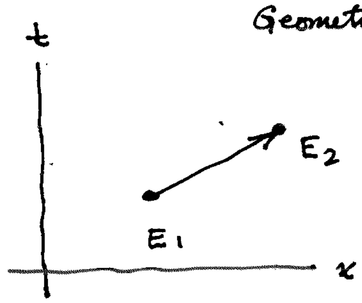
Geometry of space time $(c = 1)$.



superscript posn conventional ↓ 2/4/10

4-vectors: $\underset{\sim}{a} = (a^t, a^x, a^y, a^z)$

↑ typographical notation

Archetypical example: displacement vector in S-T.

Vector vs. Components.
primed vs. unprimed frame:

$$\left.\begin{aligned} a'^t &= \gamma(a^t - v a^x) \\ a'^x &= \gamma(a^x - v a^t) \end{aligned}\right\}, \qquad (c = 1)$$

Notation for components: $\underset{\sim}{a} = (a^0, a^1, a^2, a^3)$ $\quad a^\alpha$, $\boxed{\begin{array}{l}\alpha = 0,1,2,3 \\ i = 1,2,3\end{array}}$

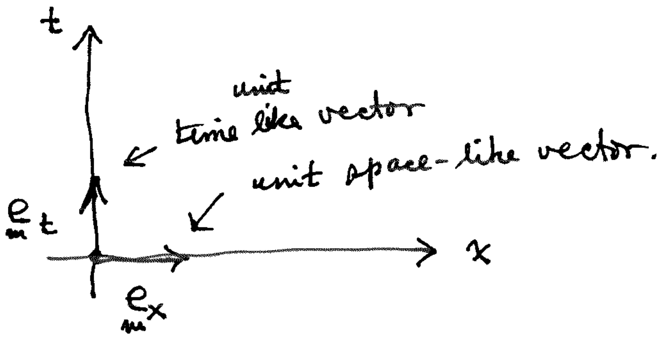
$= (a^t, \vec{a})$ $\quad \left.\begin{array}{l}\vec{a} = \text{3-vector} \\ \underset{\sim}{a} = \text{4-vector.}\end{array}\right\}$ ch. 5.

$= (a^t, a^i)$

Invariant "square len" $= -(a^t)^2 + (a^x)^2 + (a^y)^2 + (a^z)^2$
$= -(a'^t)^2 + (a'^x)^2 + (a'^y)^2 + (a'^z)^2$ same in all frames.

"Length" $= \left|\text{square len}\right|^{1/2}$ (Hartle).

Introduce unit vectors:



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$\underset{\sim}{e}_t = (1, 0, 0, 0)$

$\underset{\sim}{e}_x = (0, 1, 0, 0)$

$\underset{\sim}{e}_y = (0, 0, 1, 0)$

$\underset{\sim}{e}_z = (0, 0, 0, 1)$

components of unit vectors of unprimed frame in the unprimed frame.

(But you <u>could</u> express those components in a different frame.)

Subscript position of indices on unit vectors is conventional.
Superscript " " " " components of vectors " "
e.g. a^t, a^x

numbers.

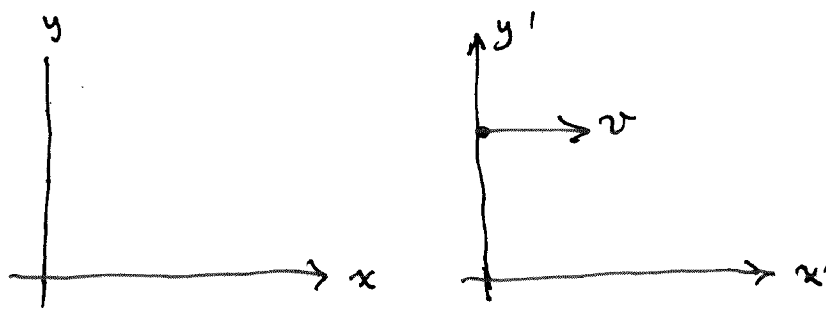
Then $$\underset{\sim}{a} = \sum_{\alpha=0}^{3} a^\alpha \underset{\sim}{e}_\alpha$$

vectors

<u>Einstein Summation Convention</u>: Omit sum on repeated indices, sum implied. So e.g. we write

$$\underset{\sim}{a} = a^\alpha \underset{\sim}{e}_\alpha.$$

How do basis vectors transform? (From one frame to another).
Consider the usual boost down x-axis:



(purely spatial picture)

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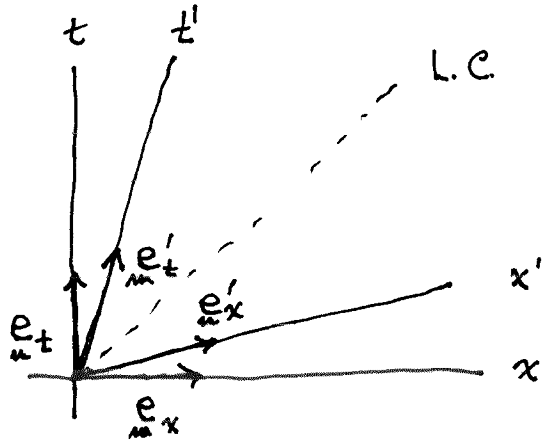


Space-time picture

what is relation between $\mathbf{e}_t, \mathbf{e}_x$ and $\mathbf{e}'_t, \mathbf{e}'_x$?

Well, if $\mathbf{a}$ is any vector, then

$$a^t = \gamma(a'^t + v\, a'^x)$$
$$a^x = \gamma(a'^x + v\, a'^t).$$

But let $\mathbf{a} = \mathbf{e}'_t$; so $a'^t = 1,\ a'^x = 0.$

then $a^t = \gamma,\ a^x = \gamma v$, so

$$\mathbf{a} = \mathbf{e}'_t = \gamma\, \mathbf{e}_t + \gamma v\, \mathbf{e}_x$$

Similarly, let $\mathbf{a} = \mathbf{e}'_x$ so $a'^t = 0,\ a'^x = 1.$ Then we find

$$\mathbf{e}'_x = \gamma v\, \mathbf{e}_t + \gamma\, \mathbf{e}_x .$$

In summary:

$$\boxed{\begin{aligned} \mathbf{e}'_t &= \gamma(\mathbf{e}_t + v\, \mathbf{e}_x) \\ \mathbf{e}'_x &= \gamma(v\, \mathbf{e}_t + \mathbf{e}_x) \end{aligned}}$$

Compare to transf. law for <u>components</u> of a vector:

$$\left.\begin{aligned} a'^t &= \gamma(a^t - v\, a^x) \\ a'^x &= \gamma(-v a^t + a^x) \end{aligned}\right\}$$

different matrix.

→ usual Lorentz transf.

3. Scalar products of vectors.

Define $\underset{\sim}{a} \cdot \underset{\sim}{b} = -(a^t)(b^t) + (a^x)(b^x) + (a^y b^y) + (a^z b^z)$.

Then $\overset{\text{invariant}}{\text{square len}}$ of $\underset{\sim}{a} = \underset{\sim}{a} \cdot \underset{\sim}{a}$.

Is the scalar product of 2 different vectors also invariant?

$$(\underset{\sim}{a} + \underset{\sim}{b})^2 = \underset{\sim}{a} \cdot \underset{\sim}{a} + 2\,\underset{\sim}{a} \cdot \underset{\sim}{b} + \underset{\sim}{b} \cdot \underset{\sim}{b}$$

all invariant

must also be invariant.

A: Yes.

Now expand in terms of unit vectors.

$$\underset{\sim}{a} = a^\alpha \underset{\sim}{e}_\alpha \qquad \text{(implied sums on } \alpha, \beta\text{)}.$$

$$\underset{\sim}{b} = b^\beta \underset{\sim}{e}_\beta$$

$$\underset{\sim}{a} \cdot \underset{\sim}{b} = a^\alpha b^\beta (\underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta).$$

what are $\underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta$?

Just work them out:

$$\underset{\sim}{e}_\alpha \cdot \underset{\sim}{e}_\beta = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \equiv \eta_{\alpha\beta}$$

Defines metric tensor $\eta_{\alpha\beta}$.

This has same form in all Lorentz (inertial) frames, i.e.

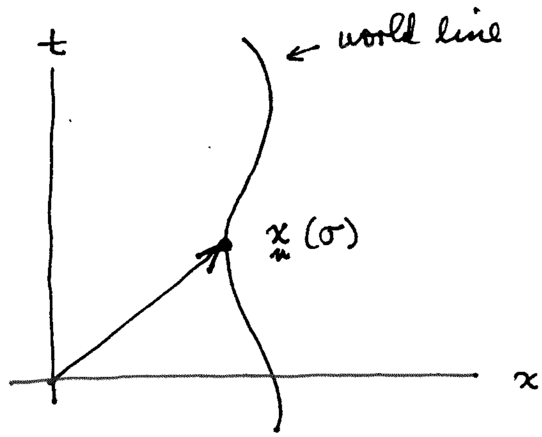
$$\mathbf{e}'_\alpha \cdot \mathbf{e}'_\beta = \eta_{\alpha\beta} = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta .$$

Then scalar product becomes

$$\boxed{\mathbf{a} \cdot \mathbf{b} = a^\alpha b^\beta \eta_{\alpha\beta} = a'^\alpha b'^\beta \eta_{\alpha\beta}} \qquad \text{(any Lorentz frame)}$$

4. Principle of relativity requires that laws of physics have same form in all inertial frames. We accomplish this by expressing the laws of physics in terms of 4-vectors and their scalar products, i.e. geometrically using the Minkowski geometry of spacetime.

5. Suppose we have an accelerated particle. Let σ be an arbitrary parameter of its world line, which can be represented by a 4-vector $\mathbf{x} = \mathbf{x}(\sigma)$.



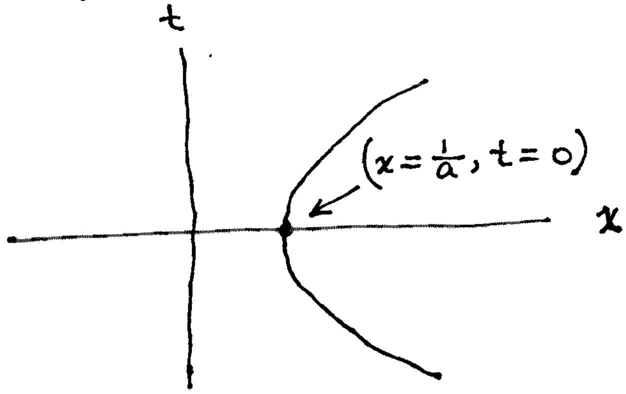
$$\mathbf{x} = (x^0, x^1, x^2, x^3)$$
$$= (t, x, y, z).$$

The parameter σ can be anything convenient. Sometimes proper time τ is the most convenient.

<u>An example:</u> Let

$$x(\sigma) = \frac{1}{a}\cosh\sigma$$
$$t(\sigma) = \frac{1}{a}\sinh\sigma.$$

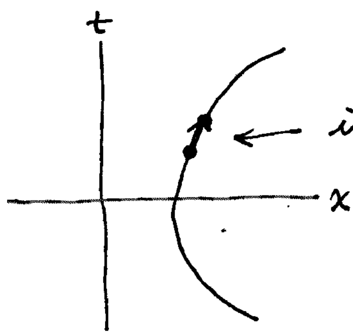
(a = a parameter). Plot world line. It is an hyperbola.



We can eliminate σ to get an equation of the world line:

$$x^2 - t^2 = \frac{1}{a^2}\left(\cosh^2\sigma - \sinh^2\sigma\right) = \frac{1}{a^2}.$$ std. form of hyperbola

What about proper time? Consider 2 close points on world line



infinitesimal displacement $d\underset{\sim}{x} = (dt, dx)$.

Then $$dx = \frac{1}{a}\sinh\sigma\, d\sigma$$
$$dt = \frac{1}{a}\cosh\sigma\, d\sigma$$

$$d\tau^2 = dt^2 - dx^2 = \frac{1}{a^2}\left(\cosh^2\sigma - \sinh^2\sigma\right) d\sigma^2 = \frac{d\sigma^2}{a^2}$$

So $d\sigma = a\, d\tau$, $\sigma = a\tau$ if we assume $\sigma = 0$ @ $\tau = 0$.

elim. σ in favor of τ and
So we can parameterize world line in terms of τ:

$$\left.\begin{aligned} x &= \frac{1}{a}\cosh a\tau \\ t &= \frac{1}{a}\sinh a\tau \end{aligned}\right\}$$

Then
$$\left.\begin{aligned} \frac{dx}{d\tau} &= \sinh a\tau \\ \frac{dt}{d\tau} &= \cosh a\tau \end{aligned}\right\}$$
← These are components of the world velocity of the particle.

6. World velocity

Generally, define $\boxed{\underset{\sim}{u} = \dfrac{d\underset{\sim}{x}}{d\tau}}$

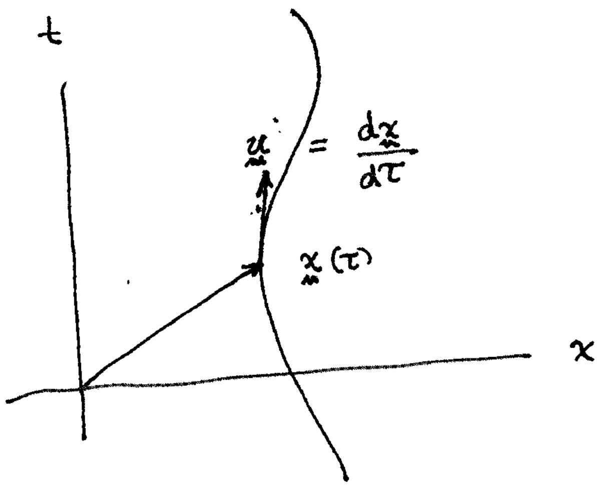
$d\underset{\sim}{x}$ is a 4-vector, because it is a displacement in S-T. And $\underset{\sim}{u}$ is a 4-vector because $d\tau$ is invariant.

Contrast with usual 3-velocity $\vec{v} = \dfrac{d\vec{x}}{dt}$. This cannot be the spatial part of a 4-vector because, while $d\vec{x}$ is the spatial part of the 4-vector $\underset{\sim}{x}$, dt is not invariant. Therefore we expect the world velocity to be more useful in relativity than the ordinary 3-velocity.

But there is a relation between $\underset{\sim}{u}$ and $\vec{v}$.

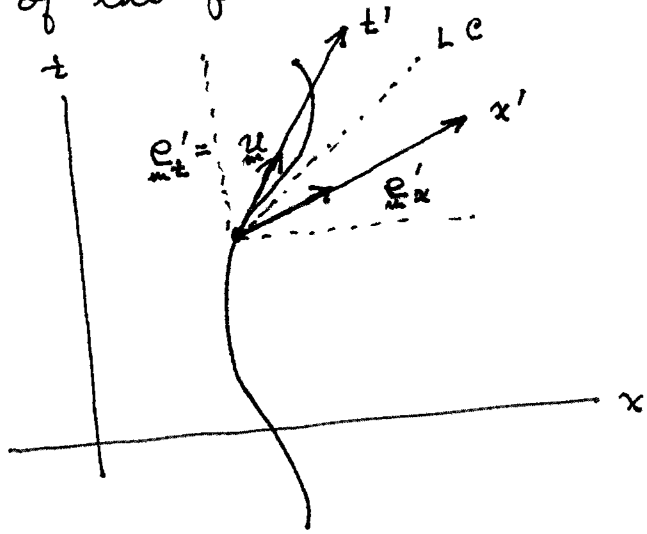
Consider
$$\underset{\sim}{u}\cdot\underset{\sim}{u} = \frac{d\underset{\sim}{x}\cdot d\underset{\sim}{x}}{d\tau^2} = \frac{-dt^2+dx^2+dy^2+dz^2}{d\tau^2} = \underbrace{-\frac{d\tau^2}{d\tau^2}}_{=-1}$$

Thus $\boxed{\underset{\sim}{u}\cdot\underset{\sim}{u} = -1}$ The world velocity is a unit time-like vector, tangent to the world line:



$\underset{\sim}{u} \cdot \underset{\sim}{u} = -1.$

In fact, $\underset{\sim}{u}$ is the unit vector along the t'-axis of a frame in which the particle is instantaneously at rest (the "rest frame" of the particle)



Since $\underset{\sim}{u} \cdot \underset{\sim}{u} = -\left(\frac{dt}{d\tau}\right)^2 + \left(\frac{d\vec{x}}{d\tau}\right)^2 = -1,$

i.e., $d\tau^2 = dt^2 - |d\vec{x}|^2,$ we have (divide by dt^2)

$$\left(\frac{d\tau}{dt}\right)^2 = 1 - \left(\frac{d\vec{x}}{dt}\right)^2 = 1 - v^2 = \frac{1}{\gamma^2},$$

or $\boxed{\frac{dt}{d\tau} = \gamma}$. We knew this already, it is the time dilation, $d\tau = \frac{dt}{\gamma}$.

As for the spatial part of $\underset{\sim}{u}$,

$$\frac{d\vec{x}}{d\tau} = \frac{dt}{d\tau}\frac{d\vec{x}}{dt} = \gamma\vec{v}.$$

Thus

$$\boxed{\underset{\sim}{u} = (u^0, \vec{u}) = (\gamma, \gamma\vec{v})}$$

Expresses world velocity in terms of ordinary velocity.

7. World acceleration

Similarly, we can go on and define the world acceleration:

$$\underset{\sim}{a} = \frac{d\underset{\sim}{u}}{d\tau} = \frac{d^2\underset{\sim}{x}}{d\tau^2}.$$

The world acceleration $\underset{\sim}{a}$ can be expressed in terms of the ordinary 3-velocity $\vec{v}$ and 3-acceleration $\frac{d\vec{v}}{dt}$, but the expressions start to get messy.

Important property of world accel. Take $\underset{\sim}{u}\cdot\underset{\sim}{u} = -1$ and apply $\frac{d}{d\tau}$:

$$\frac{d}{d\tau}(\underset{\sim}{u}\cdot\underset{\sim}{u}) = \underset{\sim}{a}\cdot\underset{\sim}{u} + \underset{\sim}{u}\cdot\underset{\sim}{a} = 2\,\underset{\sim}{u}\cdot\underset{\sim}{a} = 0.$$

$$\boxed{\underset{\sim}{u}\cdot\underset{\sim}{a} = 0}$$

The world acceleration is Minkowski orthogonal to the world velocity. That means it is space-like, in fact, it is purely spatial in the rest frame of the particle.

Go back to our example:

$$x = \frac{1}{a}\cosh a\tau \quad\Big|\quad \frac{dx}{d\tau} = \sinh a\tau \quad\Big|\quad \frac{d^2x}{d\tau^2} = a\cosh a\tau$$

$$t = \frac{1}{a}\sinh a\tau \quad\Big|\quad \frac{dt}{d\tau} = \cosh a\tau \quad\Big|\quad \frac{d^2t}{d\tau^2} = a\sinh a\tau.$$

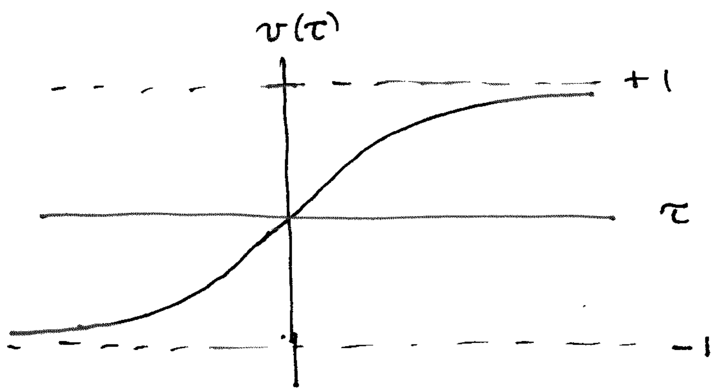
Notice, $\underset{\sim}{a}\cdot\underset{\sim}{a} = -a^2\sinh^2 a\tau + a^2\cosh^2 a\tau = a^2$.

The magnitude of the 4-acceleration is constant. Does this mean physically that the particle is uniformly accelerated? (Later).

Let's do at least this: Let's transform those vectors $\underset{\sim}{u}, \underset{\sim}{a}$ the rest frame of the particle.

Notice from above,

$$v = \frac{dx}{dt} = \text{velocity of particle} = \frac{\frac{dx}{d\tau}}{\frac{dt}{d\tau}} = \tanh a\tau.$$



$v \to \pm c$

as $\tau \to \pm\infty$.

Also, $\gamma = \dfrac{dt}{d\tau} = \cosh a\tau$.

1st. transform $\underset{\sim}{u}$ to rest frame.

$$u^t = \gamma, \quad u^x = \gamma v$$

$$u'^t = \gamma(u^t - v u^x) = \gamma(\gamma - \gamma v^2) = \gamma^2(1-v^2) = 1$$

$$u'^x = \gamma(u^x - v u^t) = \gamma(\gamma v - \gamma v) = 0.$$

$\underset{\sim}{u} = (1, 0)$ in rest frame: it is $\underset{\sim}{e}'_t$ as remarked above.

Now $\underset{\sim}{a}$:

$$a'^t = \gamma(a^t - v a^x) = \cosh a\tau \left(a \,\cancel{\cosh}\, \overset{\sinh}{} a\tau - \tanh a\tau \, a \cosh a\tau\right) = 0$$

$$a'^x = \gamma(a^x - v a^t) = \cosh a\tau\left(a \cosh a\tau - (\tanh a\tau)(a \sinh a\tau)\right)$$

$$= a(\cosh^2 a\tau - \sinh^2 a\tau) = a.$$

So, $\underset{\sim}{a} = (0, a)$ in rest frame.

8. ~~Mechanics~~ Energy and Momentum in Special Relativity

We have generalized Newton's first law (free particles and inertial frame) to S.R. What changes is $t \neq t'$ and the constancy of speed of light (required if principle of relativity is combined with Maxwell's equations.)

Let's now look at energy and momentum in S.R. Recall in Newtonian mech,

$$K = \tfrac{1}{2} m v^2 \quad \text{(kin. energy)}.$$

$$\vec{p} = m\vec{v} = m \frac{d\vec{x}}{dt}$$

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Start with momentum. Seems unlikely that $m \frac{d\vec{x}}{dt}$ is the correct expr. for momentum in S.R., except at low velocities where $dt \approx d\tau$, because $\frac{d\vec{x}}{dt}$ cannot be part of a 4-vector.

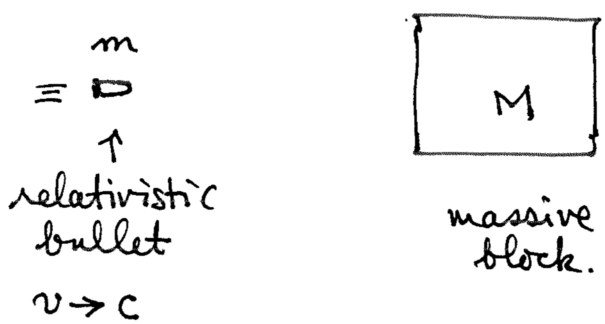
So we might guess that

$$\boxed{\vec{p} = m \frac{d\vec{x}}{d\tau} = m\gamma \frac{d\vec{x}}{dt} = m\gamma\vec{v}}$$

is the correct expression for momentum in S.R. In fact, this is right. Two points.

1. Persuasive arguments can be made showing that this is right on the basis of Maxwell's eqn (E+M) + princ. of relativity. But Hartle doesn't go into E+M.

2. We don't need theory to find the correct expression for momentum in S.R, i.e. for particles with $v \to c$. We can do it experimentally, using conservation of momentum.



After collision block is moving with velocity $v \ll c$, so its momentum is $(M+m)v$. (The N.R. formula).

In this way we find $p(v)$ for the bullet, even when $v \to c$.

Let us accept that $\vec{p} = m\gamma\vec{v}$ is the correct expression for momentum
$= m\dfrac{d\vec{x}}{d\tau}$
in S.R. This is the spatial part of a 4-vector call it $\underset{\sim}{p}$:

$$\underset{\sim}{p} = (p^0, \vec{p})$$

We must have $\quad p^0 = m\dfrac{dt}{d\tau}$ to make a 4-vector:

$$\underset{\sim}{p} = (p^0, \vec{p}) = \left(m\frac{dt}{d\tau},\ m\frac{d\vec{x}}{d\tau}\right)$$

or

$$\boxed{\underset{\sim}{p} = m\,\underset{\sim}{u}}$$

Note,
$\underset{\sim}{p} = (m\gamma, m\gamma\vec{v})$. Since $\underset{\sim}{u}\cdot\underset{\sim}{u} = -1$, we have

$$\boxed{\underset{\sim}{p}\cdot\underset{\sim}{p} = -m^2}$$

What is physical interp. of p^0? For small velocities

$$p^0 = m\gamma = \frac{m}{\sqrt{1-v^2}} = m\left(1 + \tfrac{1}{2}v^2 + \dots\right)$$

restore c's.

$$= mc^2 + \tfrac{1}{2}mv^2 + \dots$$

$= (\text{rest mass-energy}) + (\text{N.R. kin. energy}) + \dots$

So we interpret $\quad \boxed{p^0 = E = \text{total energy}}$

$\hookrightarrow$ rest mass + kinetic.

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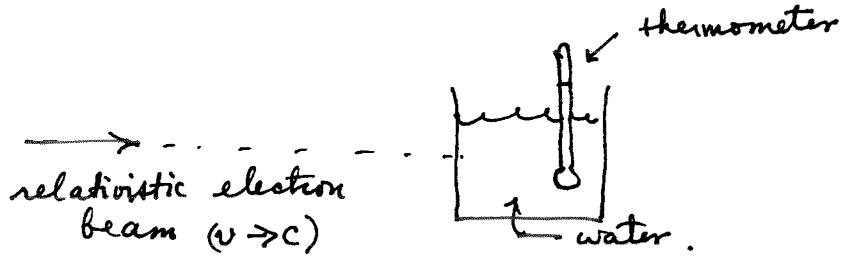
Again, we don't need theory to find the relativistic expression for kinetic energy; we can do experiments.



Measure how much heat goes into the water, you'll get K.E. of electron beam.

We define

$$\boxed{\text{kinetic energy} = E - mc^2 = mc^2(\gamma - 1).}$$

~~§1 Conservation of energy and momentum in collisions~~

9. Special case: Photons.

In relativistic expressions for energy and momentum,

$$E = m\gamma$$
$$\vec{p} = m\gamma\vec{v}$$

both $E, \vec{p} \to \infty$ as $v \to c$ (because $\gamma \to \infty$). This is for $m \neq 0$.

But because $v \to 1$ (i.e. c), $|\vec{p}| \approx E$ when the particle is relativistic.

A photon is a kind of limit of this process in which $v \to c$ but $m \to 0$ in such a way that E remains finite. In the limit, $|\vec{p}|$ becomes equal to E. So for a photon we have

$$\underset{\sim}{p} = \begin{pmatrix} E \\ \vec{p} \end{pmatrix} \quad \text{with} \quad E = |\vec{p}|.$$

This means $\underset{\sim}{p} \cdot \underset{\sim}{p} = 0$, logical since $m = 0$ and normally (for a massive particle) $\underset{\sim}{p} \cdot \underset{\sim}{p} = -m^2$. But for a photon we cannot say $\underset{\sim}{p} = m\underset{\sim}{u}$, since $\underset{\sim}{u} = \frac{d\underset{\sim}{x}}{d\tau}$ has no meaning for a photon. The reason is $d\tau = 0$ along a light-like segment. So although $\underset{\sim}{p}$ exists, $\underset{\sim}{u}$ does not.

Another point: Proper time τ cannot be used to parametrize the world line of a photon.

10. Particle collisions.

Energy and momentum are conserved in particle collisions. These laws are unified into the conservation of 4-momentum in S.R.