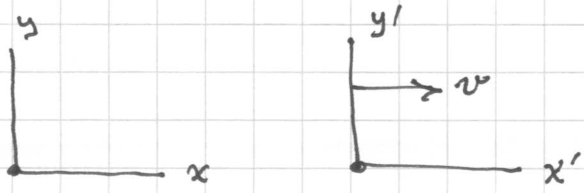
Derivation of Lorentz transformation.

Thu. 1/28/10.



"stationary" "lab" "moving"

① Principle of relativity
② Speed of light $= c$ in all frames.

Abandon Newtonian $t' = t$.

$$\begin{pmatrix} x' \\ t' \end{pmatrix} = \begin{pmatrix} \gamma & \alpha \\ \beta & \delta \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix}$$

Concentrate on x, t coords 1st.

or
$$x' = \gamma x + \alpha t$$
$$t' = \beta x + \delta t$$

① $x' = 0 \Rightarrow x = vt$

$$0 = \gamma x + \alpha t = \gamma v t + \alpha t = t(\gamma v + \alpha)$$

$$\Rightarrow \boxed{\alpha = -\gamma v}$$

$$x' = \gamma(x - vt)$$
$$t' = \beta x + \delta t$$

② $x = 0 \Rightarrow x' = -vt'$

$$x' = -\gamma v t = -v\delta t$$
$$t' = \delta t$$

$$\Rightarrow \boxed{\delta = \gamma}$$

$$x' = \gamma(x - vt)$$
$$t' = \beta x + \gamma t$$

③ $x = ct \Rightarrow x' = ct'$.

$$x' = \gamma(ct - vt) = \gamma t(c - v) = t(\gamma c - \gamma v) = t(\beta c^2 + \gamma c)$$
$$t' = \beta c t + \gamma t = t(\beta c + \gamma)$$

$$\cancel{\gamma c} - \gamma v = \beta c^2 + \cancel{\gamma c}$$

$$\boxed{\beta = -\frac{v}{c^2}\gamma.}$$

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$$\left.\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma\left(-\frac{v}{c^2}x + t\right) \end{aligned}\right\} \qquad \begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma(v)\begin{pmatrix} 1 & -v \\ -\frac{v}{c^2} & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}$$

$\gamma = \gamma(v)$.

[Swap primed, unprimed. $v \Leftrightarrow -v$]

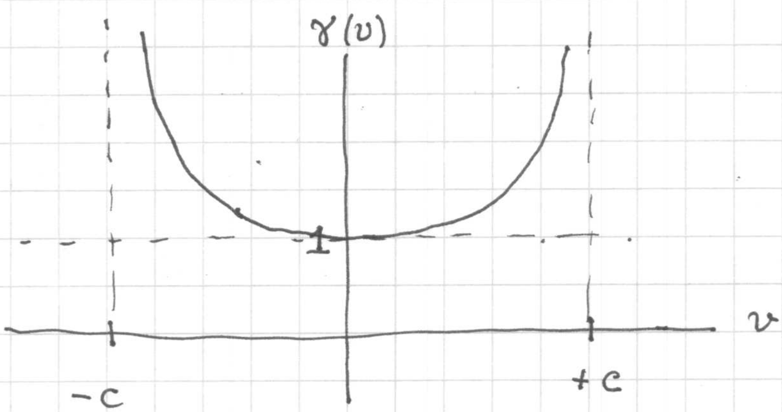
$$\begin{pmatrix} x \\ t \end{pmatrix} = \gamma(-v)\begin{pmatrix} 1 & v \\ \frac{v}{c^2} & 1 \end{pmatrix}\begin{pmatrix} x' \\ t' \end{pmatrix} = \gamma(-v)\gamma(v)\left(1 - \frac{v^2}{c^2}\right)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ t \end{pmatrix}$$

$$\gamma(v)\gamma(-v) = \frac{1}{1 - v^2/c^2}.$$

But $\gamma(v) = \gamma(-v)$

$$\boxed{\gamma(v) = \frac{1}{\sqrt{1 - v^2/c^2}}}$$

Std. notation in Special Relativity.



$\gamma(v) \to \infty$ as $v \to \pm c$.

So this gives the boost along the x-axis,

$$\boxed{\begin{aligned} x' &= \gamma(x - vt) \\ t' &= \gamma\left(t - \frac{v}{c^2}x\right) \end{aligned}}$$

what about y coordinate? Again, linearity,

$$y' = \alpha x + \beta y + \delta t$$

(not same α, β, δ as before, but 3 unknown coeffs).

But x, x' axes must coincide, otherwise picks out a preferred direc. in space, so $y=0 \Rightarrow y'=0$, ie.

$$0 = \alpha x + \delta t \quad \Rightarrow \quad \alpha = \delta = 0.$$

So,

$$\left.\begin{aligned} y' &= \beta(v)\, y \\ y &= \beta(-v)\, y' \end{aligned}\right\} \Rightarrow \quad y = \beta(v)\beta(-v)\, y$$

$$\Rightarrow \beta(v)\beta(-v) = 1.$$

But $\beta(v) = \beta(-v)$ by same logic as $\gamma(v)$ above, so $\beta(v)^2 = 1$, $\beta(v) = \pm 1$. But we must have ~~β~~ $y = y'$ when $v=0$, so $\beta(0) = +1$. By continuity, $\beta(v) = +1$ for all other values of v, and $\boxed{y'=y}$. Sim we find $\boxed{z'=z}$. The Lorentz boost does not affect distances $\perp$ to the direction of motion.

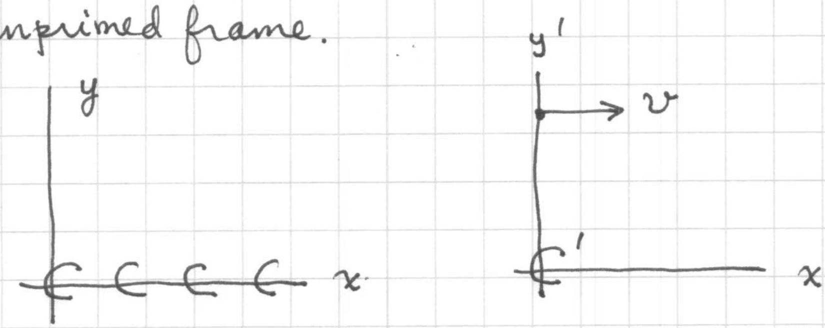
Altogether, we get:

$$\boxed{\begin{aligned} t' &= \gamma\left(t - \frac{v}{c^2}x\right) \\ x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \end{aligned}}$$

Boost down x-axis.

Consequences. I. Time dilation.

Let a clock C' be located at origin of primed coordinates. Let x axis be filled with synchronized clocks C C stationary in unprimed frame.



origins coincide at $t = t' = 0$, so both clocks read $t = 0$ $t' = 0$ when origins coincide. As primed clock moves down x-axis, we can compare it to the unprimed clocks it passes. How does t compare to t'?

Use inverse of Lorentz transf.,

$$x = \gamma(x' + vt')$$
$$t = \gamma\left(t' + \frac{v}{c^2}x'\right)$$

set $x' = 0$

get $t = \gamma t'$ or $\boxed{t' = \frac{t}{\gamma}}$ Since $\gamma \geq 1$ we have $t' \leq t$.

The moving clock runs slower.

Apparent paradox: Doesn't this violate principle of relativity? Seems that moving frame is distinguished from stationary frame.

Or: If frames are equivalent, then stationary clock must run slower when viewed in moving frame. How can this be? (Later.)

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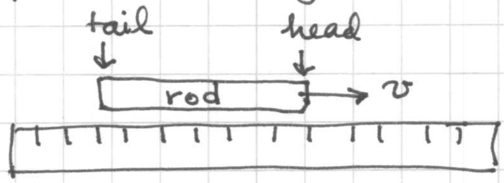
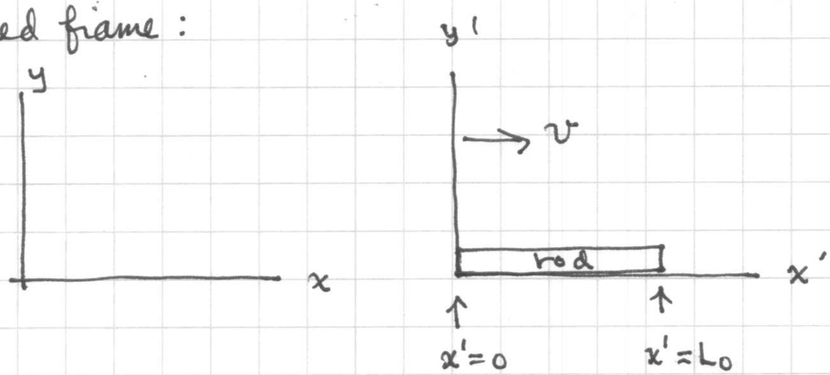
II. Lorentz contraction.

What we mean by measuring the length of a moving rod ... suppose it is moving past a stationary meter stick...



We measure the positions of the head and tail at the same time; the measuring events are simultaneous in the rest frame (of the meter stick).

Suppose the rod has "natural" length L_0, ie when measured in a frame in which the rod is stationary. Let it be stationary in the primed frame:



Again assume origins coincide at $t=t'=0$, so tail of rod is at $x=x'=0$ when $t=t'=0$. Where is head of rod (in unprimed frame) at $t=0$? Call it $x=L$, length of moving rod as seen in stationary frame.

Use $\quad x' = \gamma(x - vt) \quad$ set $\quad x' = L_0, \; x = L, \; t = 0$

get $\quad L_0 = \gamma L, \quad \boxed{L = \frac{L_0}{\gamma}} \quad \begin{matrix} L \leq L_0 \\ \gamma \geq 1 \end{matrix}$

The moving rod appears shorter than its natural length.

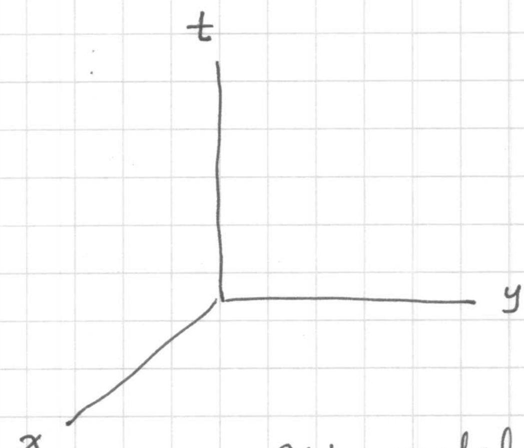
Again, apparent paradox: Doesn't this distinguish the two frames? If we believe the frames are equivalent, then rod fixed in stationary frame must appear shorter when seen from moving frame.

Resolution later.

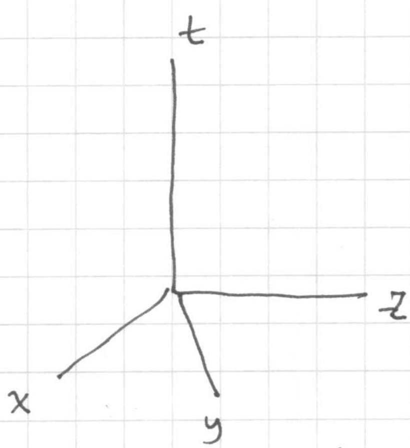
Now space-time diagrams, and spacetime.

Since $t = t'$ no longer valid in relativity theory, implies a kind of unification of space and time, what we call spacetime. If we suppress y, z, we get a 1+1 model for spacetime, with coords x, t. If we suppress just z, we get a 2+1 model with coords x, y, t. If we use the full dimensions, we get 3+1 model, but we can't sketch that.

1+1 model

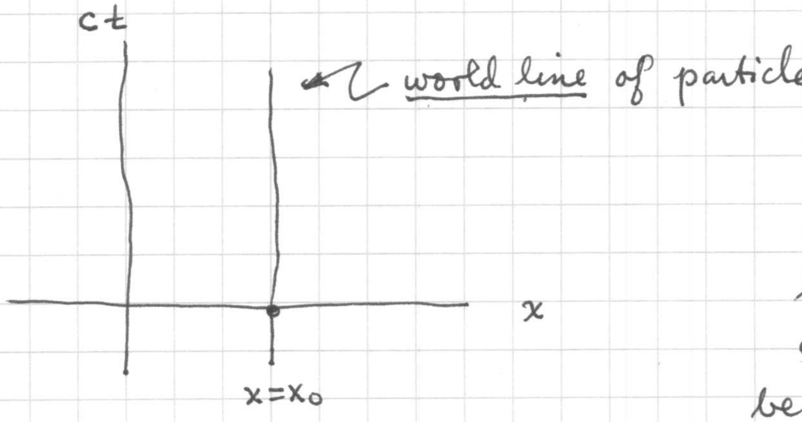


2+1 model



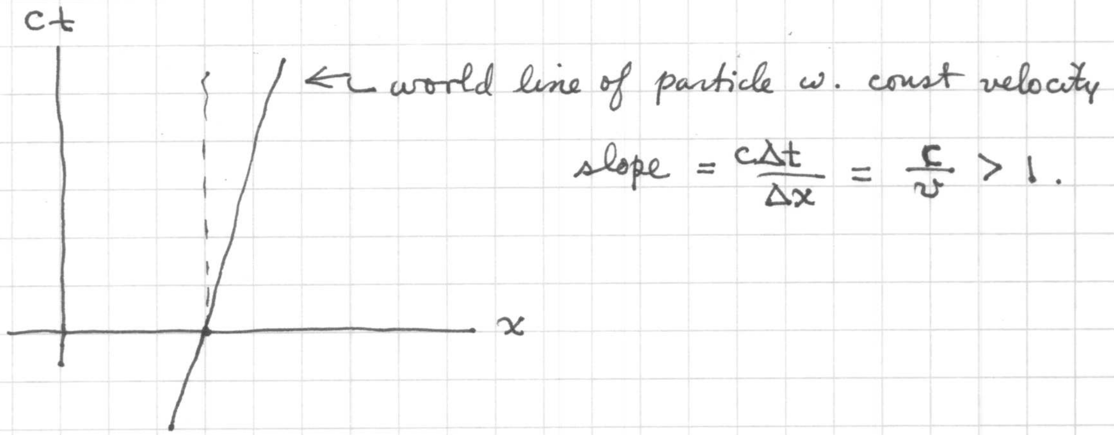
3+1 model — can't draw very well too many dimensions.

Sketch a <u>stationary particle</u> $x = x_0$ in a 1+1 diagram:



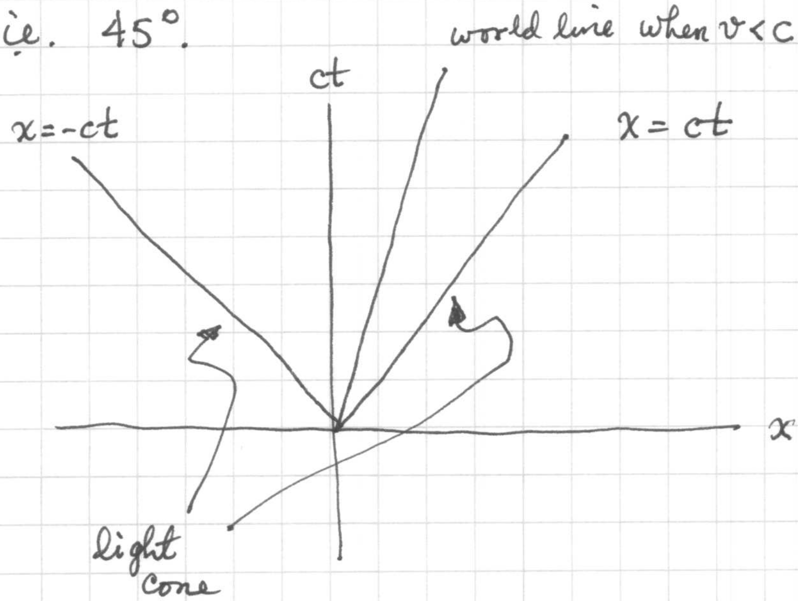
Sometimes we use ct instead of t on the vertical axis, because has same dimensions as x y z.

A particle moving at a const. velocity has a world line with some slope. The slope indicates the velocity.



$$\text{slope} = \frac{c\Delta t}{\Delta x} = \frac{c}{v} > 1.$$

Particle moving at $v = c$ (a photon) has slope $\frac{c\Delta t}{\Delta x} = \frac{c}{v} = 1$, i.e. 45°.

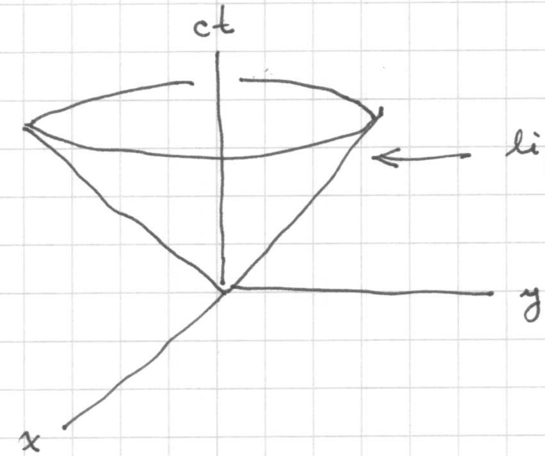


Usually we think that all particles obey $|v| \leq c$.

Their world lines lie inside lines $x = \pm ct$, defined by light pulse emitted at $x=t=0$.

These lines are called the (forward) <u>light</u> <u>cone</u>.

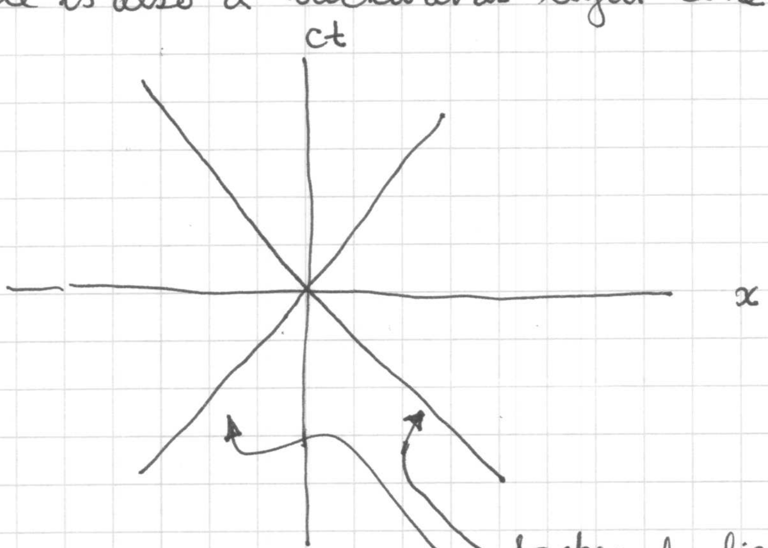
It's called a cone because if we look at the locus of points $|\vec{x}| = ct$ in 2+1 or 3+1 dimensions, it is a cone 1/28/10



light cone: $\sqrt{x^2+y^2} = ct$

Points of space time are called <u>events</u>. The forward light cone is the set of events: the light pulse emitted at $\vec{x}=0, t=0$ passes some other point $\vec{x}$ at time t. This is the set of events reached by the light pulse at $t \geq 0$. ~~There~~

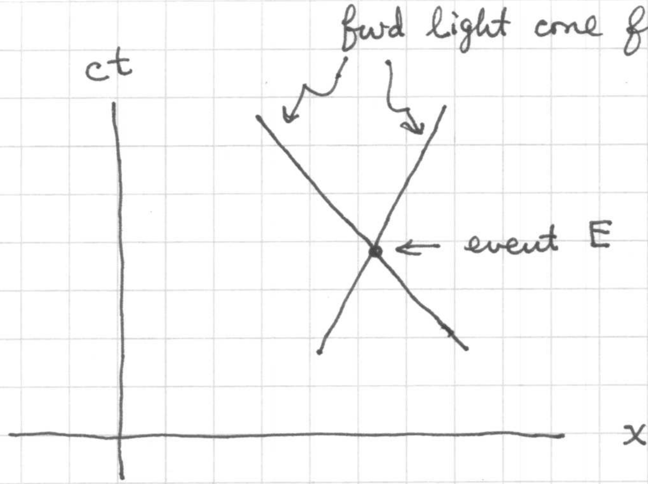
There is also a backwards light cone:



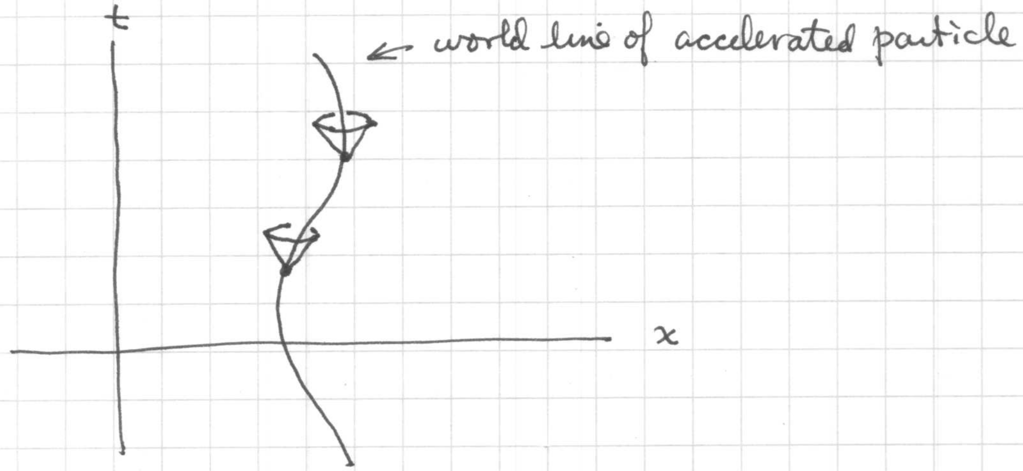
backwards light cone

The backward light cone is the set of events with $t < 0$ such that a light signal emitted at that event will reach $\vec{x} = 0$ at $t = 0$.

Light cones don't have to be based at the origin. A light pulse can be emitted at any event, and defines a light cone



An accelerated particle has a curved world line, but if $|v| < c$ it means the slope is always > 1.



This means that the world line always lies in the interior of the fwd light cone attached to any event the world line passes through. Assuming all particle velocities are $\le c$, it means that the only events that can be influenced by a given event lie inside or on the forward light cone. Such events are causally connected. Events lying outside the fwd light cone are causally disconnected from the given event. Light cones specify the causal structure of spacetime.

Here is another example of accelerated motion (circular motion) in a 2+1 diagram:

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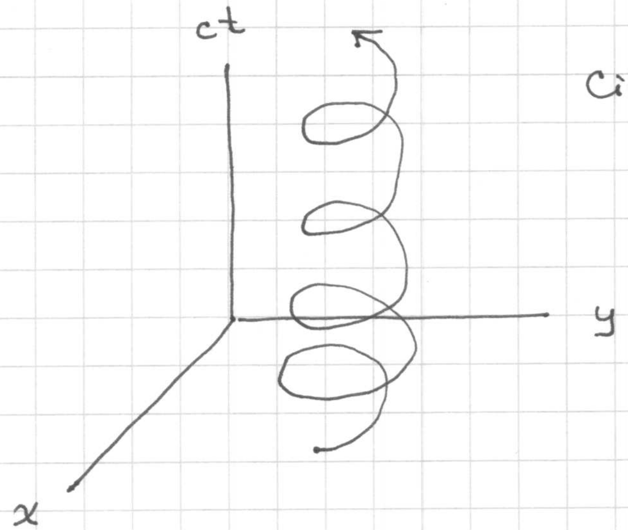
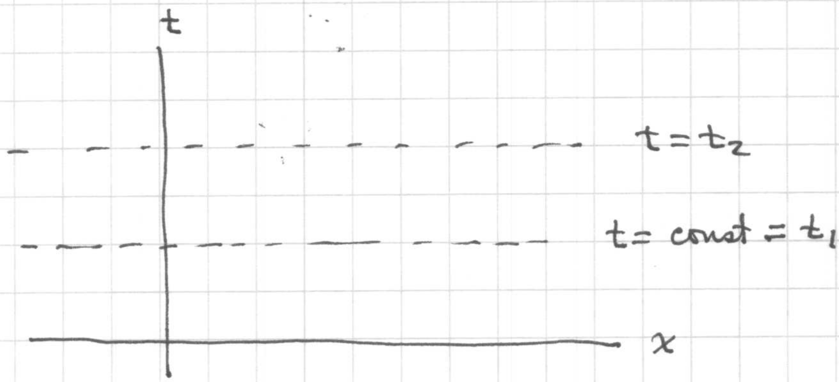


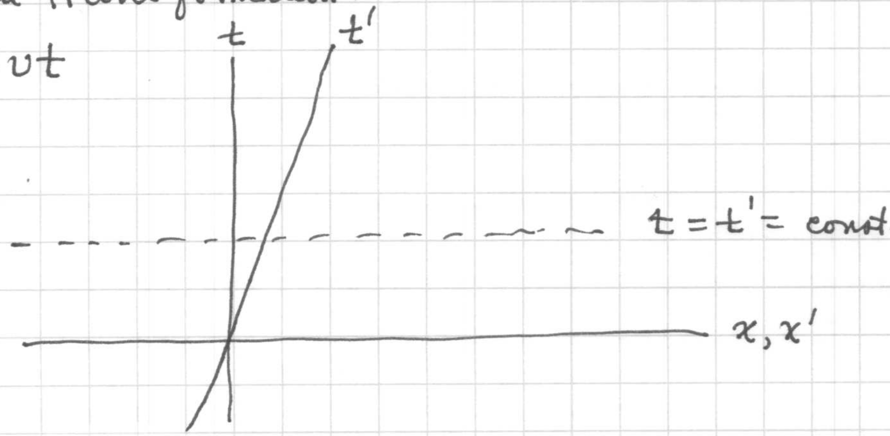
Circular motion.
World line = helix

You can use space-time diagrams in Newtonian (pre-relativistic) physics if you want to.



The set of events that are simultaneous is a line (plane, 3 space) parallel to the x-axis (xy-plane, xyz space). In Newtonian physics, all observers agree on ~~what~~ which events are simultaneous; hence they all agree on the dotted line above. You see this if you look at the Galilean transformation:

$$x' = x - vt$$
$$t' = t$$



The x, x' axes coincide because these are the loci of $t=0$, $t'=0$, which are the same.

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The space-time diagram for a Lorentz transformation is different.

$$x' = \gamma(x - vt)$$

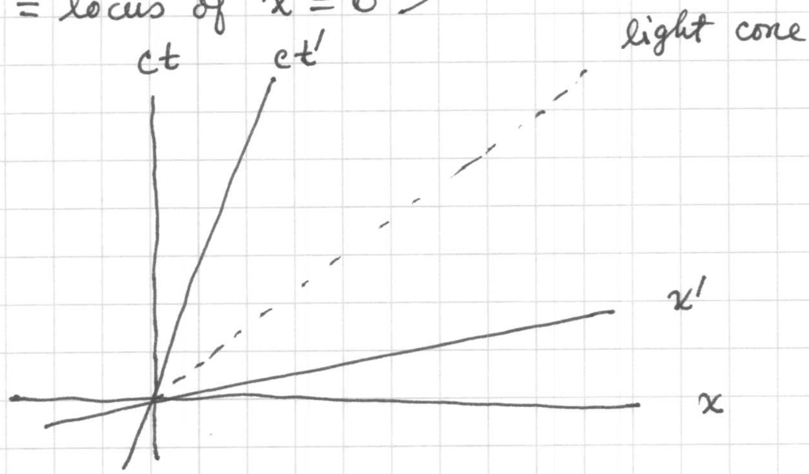
$$t' = \gamma\left(t - \frac{v}{c^2}x\right)$$

x-axis = locus of $t = 0$

x'-axis = locus of $t' = 0$, ie $x = \frac{c^2}{v} t = \left(\frac{c}{v}\right) ct$

t-axis = locus of $x = 0$

t'-axis = locus of $x' = 0$ i.e., $x = vt = \left(\frac{v}{c}\right) ct$.



The x' and ct' axes as seen in the unprimed frame collapse around the light line $x = ct$. The x' and ct' axes are not orthogonal in the diagram because the geometry (Euclidean) of the paper is not the same as the geometry of spacetime.

The ct'-axis is the world line of the origin of the primed frame.