Symplectic Reduction

in Semiclassical Mechanics

and Spin Networks

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A Dictionary $\quad \mathrm{QM} \leftrightarrows \mathrm{CM}$

Quantum	Classical
Hilbert Space $\mathcal{H}$	Phase Space or Symplectic Manifold.
	Φ or (Φ, ω) $\quad \dim \Phi = 2n$
	Coordinates (q_i, p_i)
	Symplectic Forms
	$\theta = \sum_i p_i \, dq_i, \quad \omega = d\theta = \sum_i dp_i \wedge dq_i$
Planck Rule :	Poisson Bracket
$\dim(\mathcal{H}) = \dfrac{\mathrm{vol}(\Phi)}{(2\pi\hbar)^n}$	$\{A, B\} = \sum_i \left(\dfrac{\partial A}{\partial q_i} \dfrac{\partial B}{\partial p_i} - \dfrac{\partial A}{\partial p_i} \dfrac{\partial B}{\partial q_i} \right)$

Operators and Symbols

Quantum	Classical
The quantum Hamiltonian	The classical Hamiltonian
$\hat{H} : \mathcal{H} \to \mathcal{H}$	$H = H(q, p)$
Equations of Motion	Classical Eqns. of Motion
$i\hbar \frac{\partial}{\partial t} \lvert\psi\rangle = \hat{H} \lvert\psi\rangle$	$X = \text{flow vector} = \omega^{-1} dH$

Other Operators

$$\underset{\uparrow\ \text{It's Weyl symbol}}{A(x,p)} = \int ds\, e^{-ips/\hbar} \langle x+s/2 | \underset{\uparrow\ \text{operator}}{\hat{A}} | x-s/2 \rangle \qquad (\text{Weyl transform})$$

WKB Theory in One Dimension

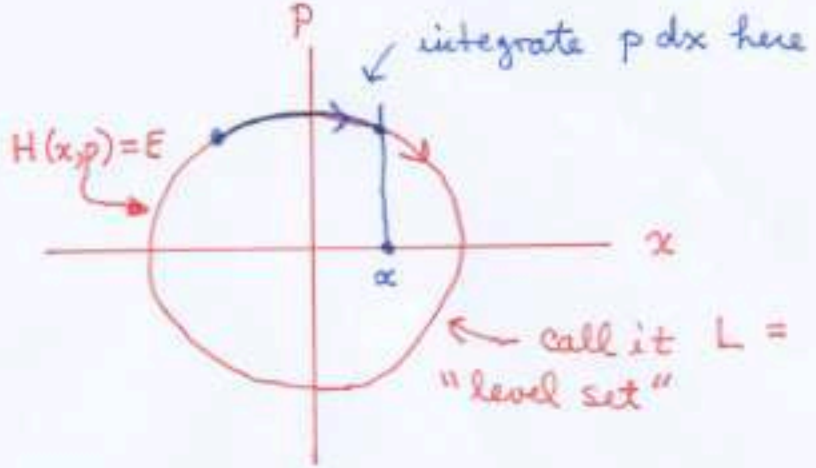
$$\hat{H}|\psi\rangle = E|\psi\rangle$$

Wave Function

$$\psi(x) = \langle x|\psi\rangle$$

Weyl Symbol = Classical Ham. $H = H(x,p)$

Level Set $H = E = H(x,p)$:



$$S(x) = \int_L \theta = \int_L^x p\,dx$$

phase of WKB wave function

$$\psi(x) = \Omega(x)\, e^{iS(x)/\hbar}$$

Multidimensional, Integrable Systems

$$\hat{H}|\psi\rangle = E|\psi\rangle$$
$$\hat{A}_2|\psi\rangle = a_2|\psi\rangle$$
$$\vdots$$
$$\hat{A}_n|\psi\rangle = a_n|\psi\rangle$$

Define
$$\begin{pmatrix} \hat{A}_1 = \hat{H} \\ a_1 = E \end{pmatrix}$$

Example:
(H, L^2, L_z)
central force.

Have set $\{\hat{A}_1, \dots, \hat{A}_N\}$ of commuting observables:

$$[\hat{A}_i, \hat{A}_j] = 0$$

Wave function

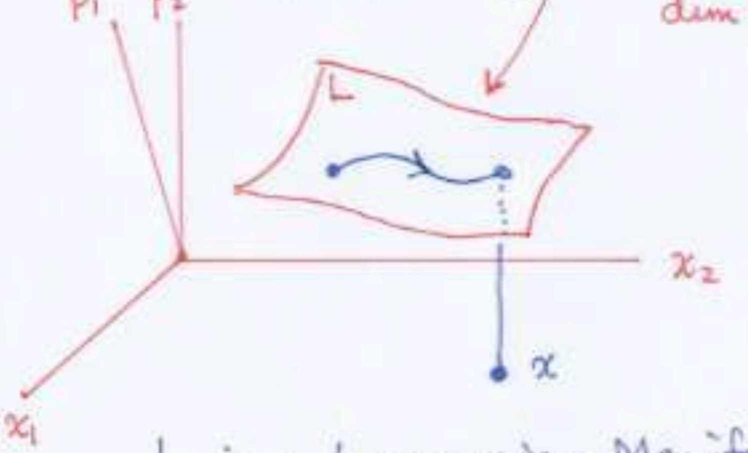
$$\psi_{a_1 \dots a_n}(x_1, \dots, x_n) = \langle x_1 \dots x_n | a_1 \dots a_n \rangle$$

$\psi_a(x) = \langle x | a \rangle$ for short

Simultaneous Level Set,

$$\left.\begin{matrix} A_1(x,p) = a_1 \\ \vdots \\ A_n(x,p) = a_n \end{matrix}\right\} \text{ call it } L$$

$\dim L = n = \frac{1}{2} \dim \Phi$



L is a Lagrangian Manifold. Means $\omega|_L = 0$, or $\int_L p\,dx$ is indep. of path.

$$S(x) = \int_L^x p\,dx = \text{phase of WKB wave fn.}$$

here

$$\psi(x) = \sum_{\text{branches}} \Omega(x)\, e^{iS(x)/\hbar}$$

What About Angular Momentum?

Carrier Space $\mathcal{C}_j$

$$j = 0, 1/2, 1, 3/2, \ldots$$

$$\mathcal{C}_j = \text{span}\{ |jm\rangle, \ m = -j, \ldots, +j \}$$

$$\dim \mathcal{C}_j = 2j+1$$

Angular Momentum Operators $\hat{J}_i$, $i = 1,2,3$

$$[\hat{J}_i, \hat{J}_j] = i\, \varepsilon_{ijk}\, \hat{J}_k$$

$$\hat{J}^2 |jm\rangle = j(j+1) |jm\rangle$$

$$J_z |jm\rangle = m |jm\rangle$$

Rotation Operators $U(g)$, $g \in SU(2)$

act on $\mathcal{C}_j$

$$U(g) |jm\rangle = \sum_{m'} |jm'\rangle\, D^j_{m'm}(g) \leftarrow \text{Wigner } D\text{-matrix}$$

What is Phase Space?

Must have

$$\frac{\text{Area}}{2\pi\hbar} = 2j+1 = \text{finite.}$$

?

Cannot be x-p plane.

Schwinger – Bargmann Model

Hilbert Space = $L^2(\mathbb{R}^2) = \{\psi(x_1,x_2)\}$

Two Harmonic Oscillators

$$\hat{H} = \tfrac{1}{2}(\hat{x}_1^2+\hat{p}_1^2) + \tfrac{1}{2}(\hat{x}_2^2+\hat{p}_2^2) = \hat{H}_1+\hat{H}_2$$

$$\hat{I} = \tfrac{1}{2}(\hat{H}-1)$$

Creation - Annihilation Operators

$$\hat{a}_\mu = \frac{\hat{x}_\mu + i\hat{p}_\mu}{\sqrt{2}}, \quad \hat{a}_\mu^+ = \frac{\hat{x}_\mu - i\hat{p}_\mu}{\sqrt{2}}, \quad \mu = 1,2$$

$$\hat{I} = \frac{1}{2}\sum_{\mu=1}^{2} \hat{a}_\mu^+ \hat{a}_\mu$$

Phase Space = $\mathbb{R}^4$, $(x_1\, x_2\, p_1\, p_2)$

$$H = \tfrac{1}{2}(x_1^2+p_1^2) + \tfrac{1}{2}(x_2^2+p_2^2)$$

$$I = \tfrac{1}{2}H$$

Complex Coordinates

$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2$

= "spinor"

$$z_\mu = \frac{x_\mu + ip_\mu}{\sqrt{2}}, \quad \bar{z}_\mu = \frac{x_\mu - ip_\mu}{\sqrt{2}}$$

$$I = \frac{1}{2}\sum_{\mu=1}^{2} \bar{z}_\mu z_\mu = \frac{1}{2}\sum_{\mu=1}^{2} |z_\mu|^2$$

$$= \tfrac{1}{4}(x_1^2+x_2^2+p_1^2+p_2^2)$$

$$\omega = \sum_{\mu=1}^{2} dp_\mu \wedge dx_\mu = -i\sum_{\mu=1}^{2} dz_\mu \wedge d\bar{z}_\mu$$

The Schwinger Oscillator

$$\hat{H} = \tfrac{1}{2}(\hat{x}_1^2 + \hat{p}_1^2) + \tfrac{1}{2}(\hat{x}_2^2 + \hat{p}_2^2)$$

$$\hat{I} = \tfrac{1}{2}(\hat{H} - 1)$$

$$\hat{H}|n_1 n_2\rangle = E_{n_1 n_2}|n_1 n_2\rangle$$

$$\hat{I}|n_1 n_2\rangle = I_{n_1 n_2}|n_1 n_2\rangle$$

$$E_{n_1 n_2} = (n_1 + 1/2) + (n_2 + 1/2) = n_1 + n_2 + 1$$

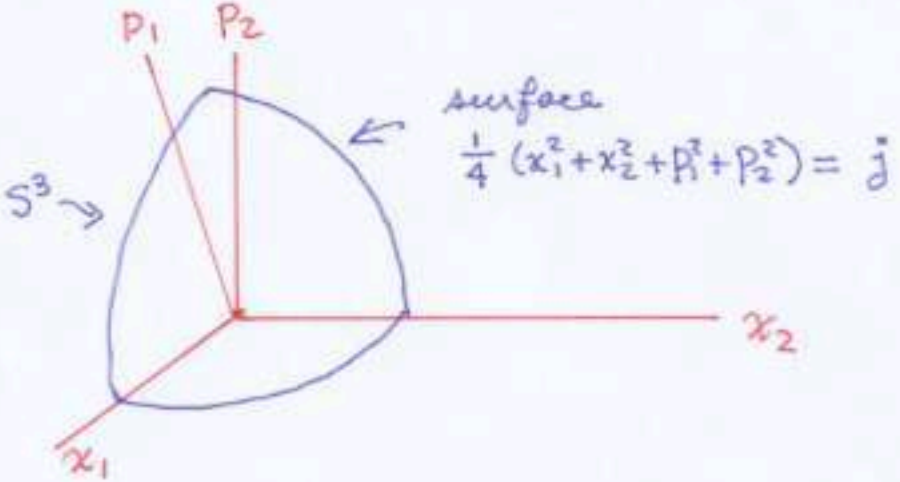
$$I_{n_1 n_2} = \tfrac{1}{2}(n_1 + n_2) \equiv j$$

Eigenvalues, Degeneracies

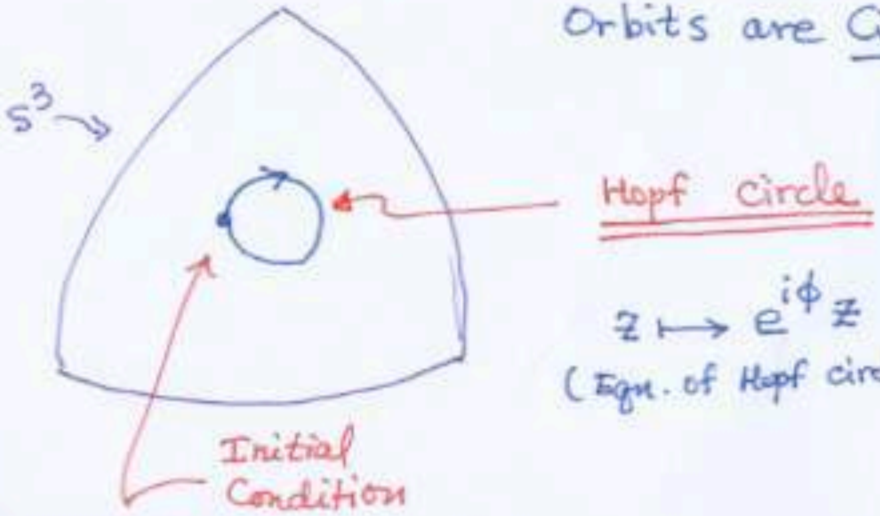
$\hat{H} = E$	1	2	3	4	5
$\hat{I} = j$	0	1/2	1	3/2	2
degen	1	2	3	4	5

degen = $2j + 1$

Level Set $H = E$ or $I = j$ is a 3-sphere S^3



Classical Orbits are Circles



$z \mapsto e^{i\phi} z$

(Eqn. of Hopf circle)

Angular Momentum in Schwinger Model

Definitions

$$\hat{J}_1 = \frac{1}{2}(\hat{x}_1\hat{x}_2 + \hat{p}_1\hat{p}_2)$$

$$\hat{J}_2 = \frac{1}{2}(\hat{x}_1\hat{p}_2 - \hat{x}_2\hat{p}_1)$$

$$\hat{J}_3 = \frac{1}{4}(\hat{x}_1^2 + \hat{p}_1^2 - \hat{x}_2^2 - \hat{p}_2^2)$$

$$\vec{\hat{J}} = \frac{1}{2}\sum_{\mu\nu} \hat{a}^+_\mu (\vec{\sigma})_{\mu\nu} \hat{a}_\nu$$

Operator Identities

$$[\hat{J}_i, \hat{J}_j] = i\varepsilon_{ijk}\hat{J}_k$$

$$\hat{J}_1^2 + \hat{J}_2^2 + \hat{J}_3^2 \equiv \vec{\hat{J}}^2 = \hat{I}(\hat{I}+1)$$

Eigenvalues:

$\hat{I} = j$, $\quad j = 0, 1/2, 1, 3/2, \ldots$

$\vec{\hat{J}}^2 = j(j+1)$

Eigenspace $I = j$ is $\mathcal{E}_j$

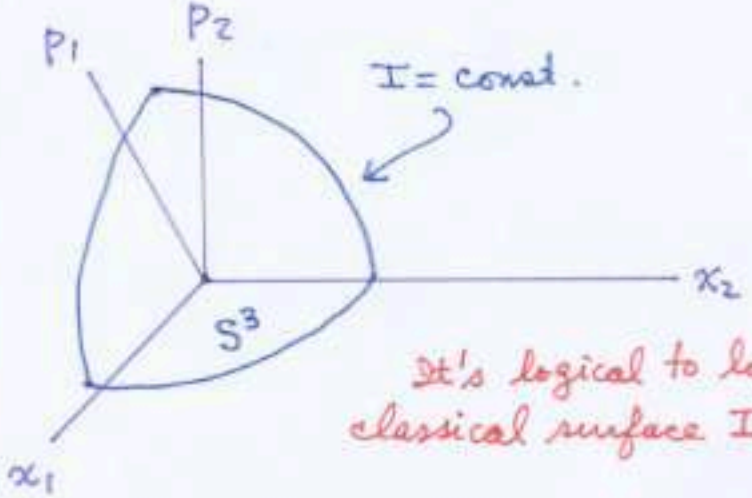
$$\boxed{\mathcal{S} = \sum_j \oplus \mathcal{E}_j}$$

$$\vec{J} = \frac{1}{2}\sum_{\mu\nu} \bar{z}_\mu (\vec{\sigma})_{\mu\nu} z_\nu = \frac{1}{2}\langle z|\vec{\sigma}|z\rangle$$

$$\{J_i, J_j\} = \varepsilon_{ijk} J_k$$

$$J^2 = I^2$$

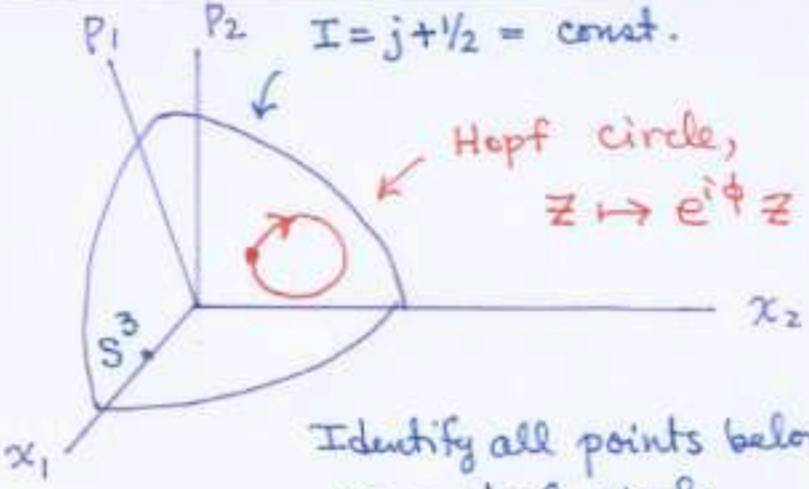
Q: What is phase space corresponding to $\mathcal{E}_j$? (Space $\hat{I} = j$).



It's logical to look at the classical surface $I = j + 1/2$.

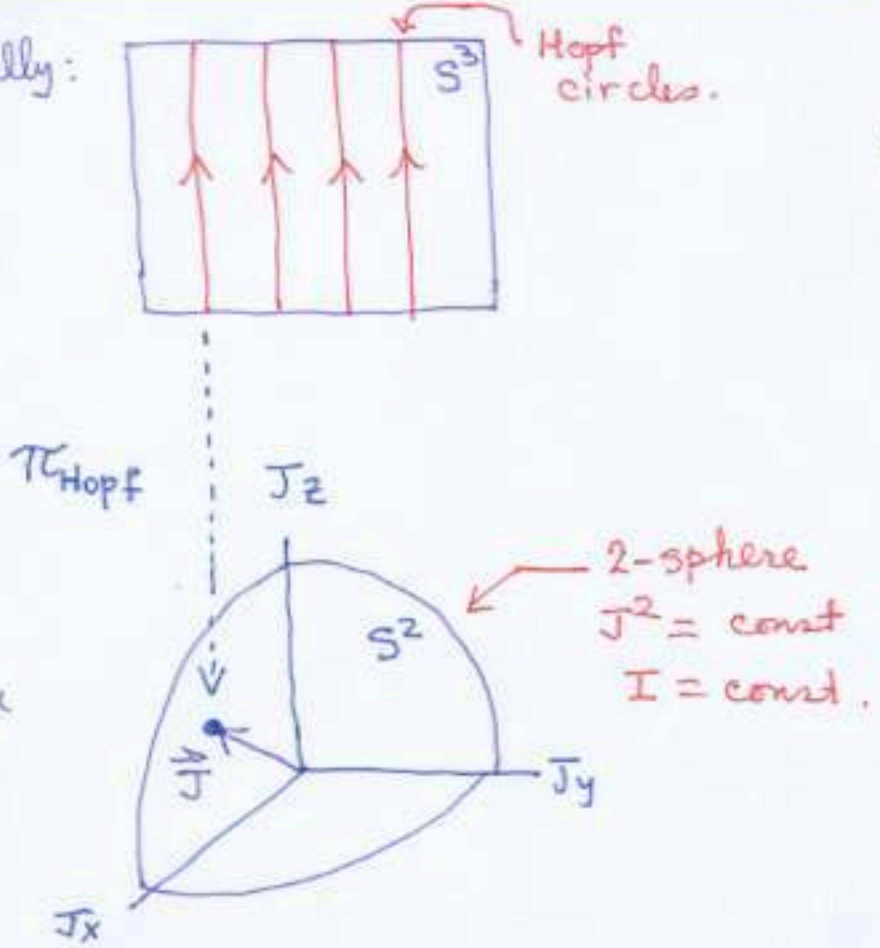
Phase Space $\Leftrightarrow \mathcal{E}_j$ is a Quotient Space

If $\hat{I} = j$ is exactly known, $\Delta I = 0$, then conjugate variable $\Delta\phi$ is completely unknown: $\Delta I \, \Delta\phi \gtrsim \hbar$



Identify all points belonging to same Hopf circle.
$(3d \to 2d)$

Schematically:



"Angular Momentum Space"

Equation of Hopf projection is the definition of $\vec{J}$:

$$\vec{J} = \frac{1}{2} \sum_{\mu\nu} \bar{z}_\mu (\vec{\sigma})_{\mu\nu} z_\nu$$

$$\pi_{\text{Hopf}} : S^3 \longrightarrow S^2$$

$$\text{or } \pi_{\text{Hopf}} : \mathbb{C}^2 \longrightarrow \mathbb{R}^3$$

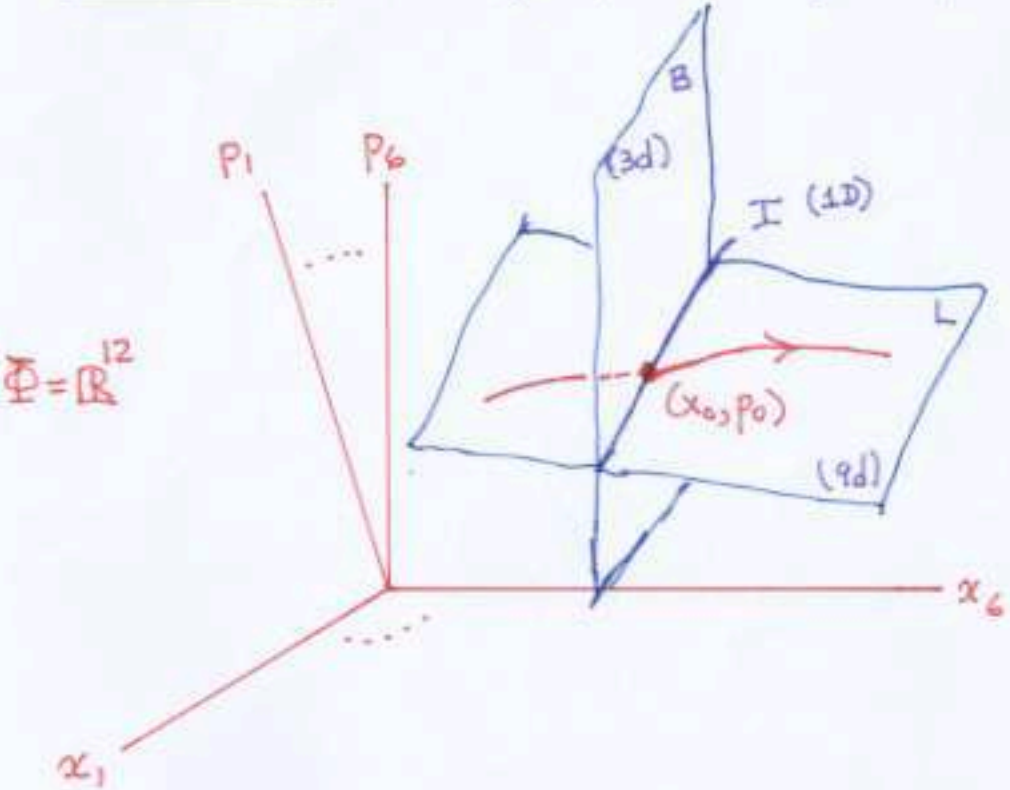
The Classical Helium Atom

$H = \sum_{i=1}^{2} \frac{p_i^2}{2m_i} + V(\vec{x}_1, \vec{x}_2)$ 6 D.O.F. $\Phi = \mathbb{R}^{12}$, coords $(\vec{x}_1, \vec{x}_2, \vec{p}_1, \vec{p}_2)$

Rotational Symmetry $\begin{pmatrix} \vec{x}_i \\ \vec{p}_i \end{pmatrix} \longmapsto \begin{pmatrix} R\vec{x}_i \\ R\vec{p}_i \end{pmatrix}, R \in SO(3)$

Generators or Momentum Map

$\vec{L} = \sum_{i=1}^{2} \vec{x}_i \times \vec{p}_i$



Level Set : $\vec{L} = \vec{L}_0$, dim L = 12 − 3 = 9

Orbit B : $R(x_0, p_0)$, $R \in SO(3)$, dim B = 3

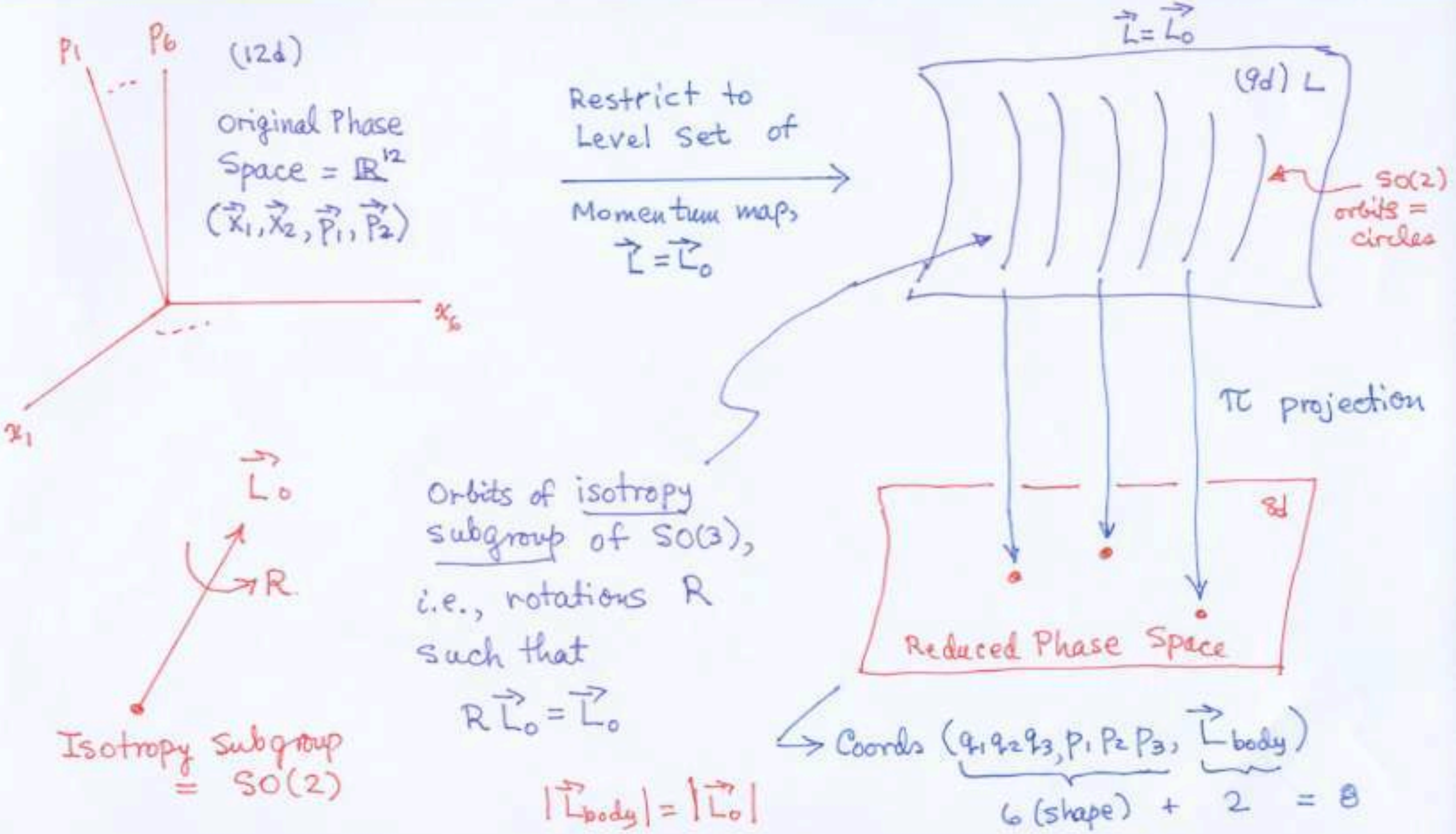
Intersection I : dim I = 1

Isotropy Subgroup

$R(\vec{L}_0, \phi)$

SO(2)

The 3-Body Reduced Phase Space



Original Phase Space = $\mathbb{R}^{12}$ $(\vec{x}_1,\vec{x}_2,\vec{p}_1,\vec{p}_2)$

Restrict to Level Set of Momentum map $\vec{L}=\vec{L}_0$

Orbits of isotropy subgroup of SO(3), i.e., rotations R such that $R\vec{L}_0=\vec{L}_0$

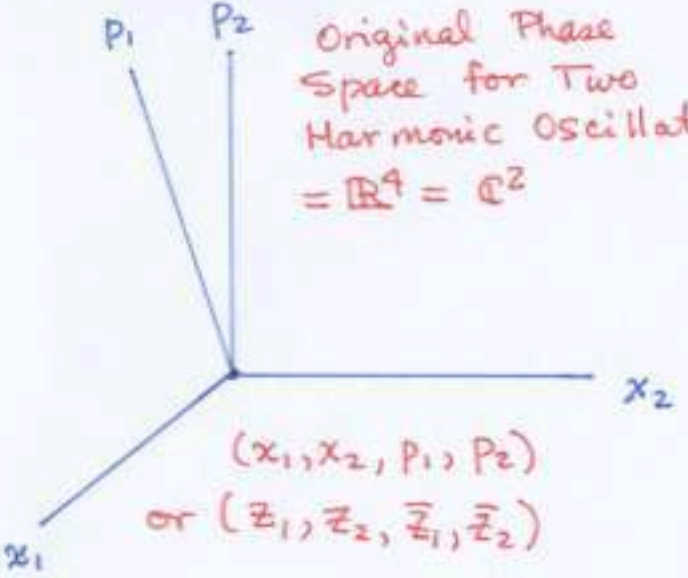
Isotropy subgroup = SO(2)

$|\vec{L}_{\text{body}}| = |\vec{L}_0|$

Coords $(\underbrace{q_1 q_2 q_3, p_1 p_2 p_3}, \underbrace{\vec{L}_{\text{body}}})$

6 (shape) + 2 = 8

Recapitulate for Hopf Fibration



Original Phase Space for Two Harmonic Oscillators $= \mathbb{R}^4 = \mathbb{C}^2$

(x_1, x_2, p_1, p_2)

or $(z_1, z_2, \bar{z}_1, \bar{z}_2)$

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \in \mathbb{C}^2$$

Restrict to Level Set $I = I_0$ of

$$\xrightarrow{\hspace{4cm}}$$

Momentum map

$$\boxed{\frac{S^3}{U(1)} = S^2 \quad \text{Hopf map}}$$

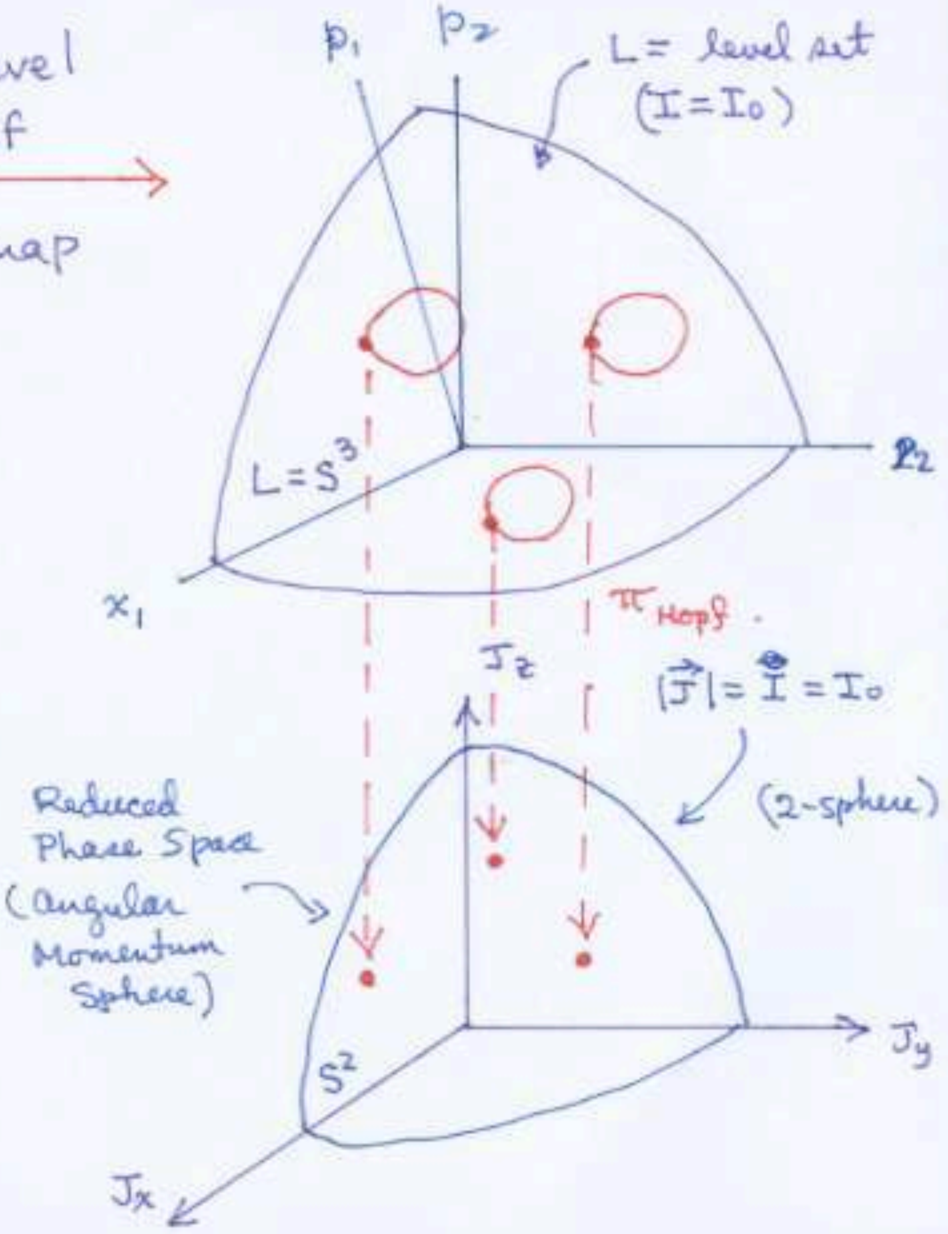
Symmetry

$$z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto e^{i\phi} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \leftarrow \text{Generated by}$$

$I = \frac{1}{2}(H_1 + H_2)$

H.O. Hamiltonian

group $U(1)$



Linear Algebra of 6j Symbol

<u>Pick</u> <u>Four</u> j's: $\{j_1\, j_2\, j_3\, j_4\}$.

Let $\mathcal{H} = \mathcal{E}_{j_1} \otimes \mathcal{E}_{j_2} \otimes \mathcal{E}_{j_3} \otimes \mathcal{E}_{j_4}$: Basis $|j_1 m_1\rangle |j_2 m_2\rangle |j_3 m_3\rangle |j_4 m_4\rangle$

$\dim(\mathcal{H}) = (2j_1+1)(2j_2+1)(2j_3+1)(2j_4+1)$

Let $\vec{J}_{tot} = \vec{J}_1 + \vec{J}_2 + \vec{J}_3 + \vec{J}_4$ (total angular momentum)

Define subspace $Z \subset \mathcal{H}$ by

$$Z = \{ |\psi\rangle \in \mathcal{H} \mid \vec{J}_{tot} |\psi\rangle = 0 \}$$

Mnemonic : Z = "zero"

States in Z are <u>invariant</u> under rotations,

$$U|\psi\rangle = |\psi\rangle \qquad U(g) = U_1(g)\, U_2(g)\, U_3(g)\, U_4(g)$$

$g \in SU(2)$

$\dim Z = \min(2j_1, 2j_2, 2j_3, 2j_4, J-2j_1, J-2j_2, J-2j_3, J-2j_4)$

$J = j_1 + j_2 + j_3 + j_4$

Basis in $\mathcal{Z}$; 6j Symbol

Vectors in $\mathcal{Z} \subset \mathcal{H}$ are <u>rotationally</u> <u>invariant</u>.

To get basis in $\mathcal{Z}$, look at eigenvectors of <u>rotationally</u> <u>invariant</u> operators, e.g. $J_{12}^2 = (\vec{J}_1 + \vec{J}_2)^2$, $J_{23}^2 = (\vec{J}_2 + \vec{J}_3)^2$, $J_{13}^2 = (\vec{J}_1 + \vec{J}_3)^2$

(or maybe volume $V = \vec{J}_1 \cdot (\vec{J}_2 \times \vec{J}_3)$)

Then 6j symbol is unitary matrix connecting these bases:

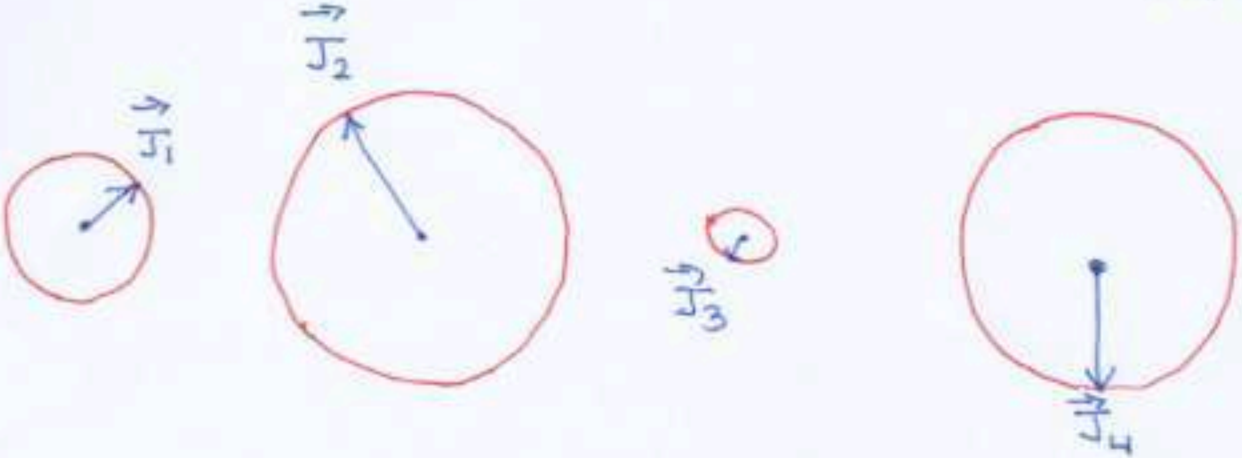
$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix} = \langle j_1 j_2 j_3 j_4 ; j_{12} | j_1 j_2 j_3 j_4 ; j_{23} \rangle$$

(idea)

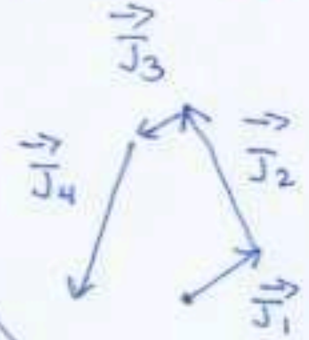
Phase Space For 4 Angular Momenta

$$\mathcal{H} = \mathbb{C}_{j_1} \otimes \mathbb{C}_{j_2} \otimes \mathbb{C}_{j_3} \otimes \mathbb{C}_{j_4}$$

$$|\vec{J}_i| = j_i + 1/2 = \text{fixed}.$$



Visualize at point of Φ:



$$\Phi = S^2_{j_1} \times S^2_{j_2} \times S^2_{j_3} \times S^2_{j_4}$$

$$\dim \Phi = 2+2+2+2 = 8$$

Symmetry: $\begin{pmatrix} \vec{J}_1 \\ \vdots \\ \vec{J}_4 \end{pmatrix} \longmapsto \begin{pmatrix} R\vec{J}_1 \\ \vdots \\ R\vec{J}_4 \end{pmatrix}$, $R \in SO(3)$

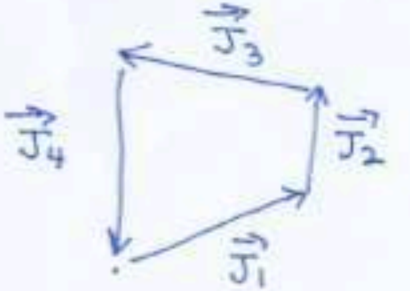
Generated by
$$\vec{J}_{tot} = \vec{J}_1 + \vec{J}_2 + \vec{J}_3 + \vec{J}_4$$
(momentum map).

Reduced Phase Space for $\mathcal{Z}$:

$\mathcal{Z} \subset \mathcal{H}$ defined by $\vec{J}_{tot}|\psi\rangle = 0$.

Semiclassically, want level set $\vec{J}_{tot} = 0$ of momentum map.

Visualize point of $\vec{L}$ as 4 angular momentum vectors forming a closed chain:



$$\dim L = \overset{\dim \Phi}{8} - \overset{\text{constraints } \vec{J}_{tot} = 0.}{3} = 5$$

But L is not the reduced phase space. Must quotient by isotropy subgroup. If $\vec{J}_{tot} = 0$, then

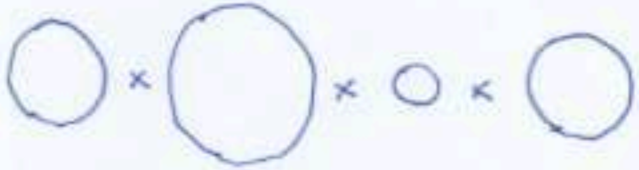
$$\vec{J}_{tot} \mapsto R\vec{J}_{tot}.$$

$R\vec{J}_{tot} = \vec{J}_{tot} = 0$ satisfied by any $R \in SO(3)$.

Isotropy subroup is whole group SO(3).

Symplectic Reduction for 6j Symbol

$\Phi = S^2 \times S^2 \times S^2 \times S^2$ (8d)



Radius $|\vec{J}_i| = j_i + 1/2$

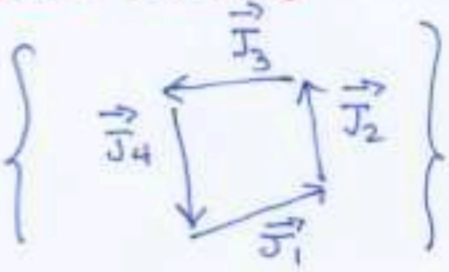
arbitrary point of Φ:

Symmetry: $\begin{pmatrix} \vec{J}_1 \\ \vdots \\ \vec{J}_4 \end{pmatrix} \longmapsto \begin{pmatrix} R\vec{J}_1 \\ \vdots \\ R\vec{J}_4 \end{pmatrix}$, $R \in SO(3)$

Momentum Map: $\vec{J}_{tot} = \vec{J}_1 + \vec{J}_2 + \vec{J}_3 + \vec{J}_4$

$$\xrightarrow[\text{(3 condition)}]{\text{Restrict to level set } \vec{J}_{tot} = 0}$$

Level set L = set of closed links :

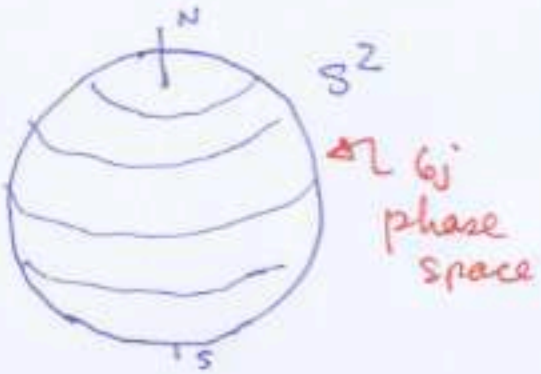


dim L = 8 − 3 = 5

(Identify closed links related by proper rotations)

$\Big\downarrow \pi^{\text{Sympl.}}_{\text{Reduc.}}$ (Ignore Orientation)

dim (reduced phase space) = 5 − 3 = 2



Can Think of D-matrices in 2 Ways

Wigner D-matrix: $D^j_{mm'}(g) = \langle jm | U(g) | jm' \rangle$ (Usual Definition)

First way: $\{ |jm\rangle,\ m = -j, \ldots, +j \}$ = basis in $\mathcal{E}_j$

Think: $U(g) : \mathcal{E}_j \rightarrow \mathcal{E}_j$ (irreducible)

Second Way: $\mathcal{E}_j \subset \mathcal{S} = L^2(\mathbb{R}^2)$ = Schwinger Hilbert Space.

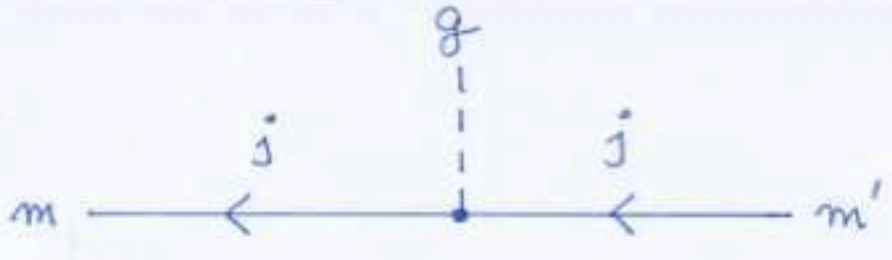
$\{ |jm\rangle ;\ j = 0, 1/2, 1, 3/2, \ldots ;\ m = -j, \ldots +j \}$ = basis in $\mathcal{S}$.

Think: $U(g) : \mathcal{S} \rightarrow \mathcal{S}$. (reducible)

Remind: $\mathcal{S} = \sum_{j=0,1/2,\ldots} \oplus\, \mathcal{E}_j$

First Way: $U(g): \mathcal{E}_j \to \mathcal{E}_j$ Spin Network

$D^j_{mm'}(g) =$



(Credit: Stedman)

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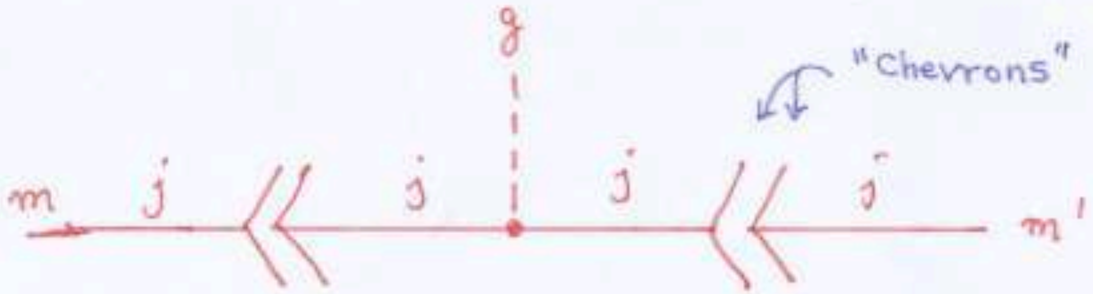
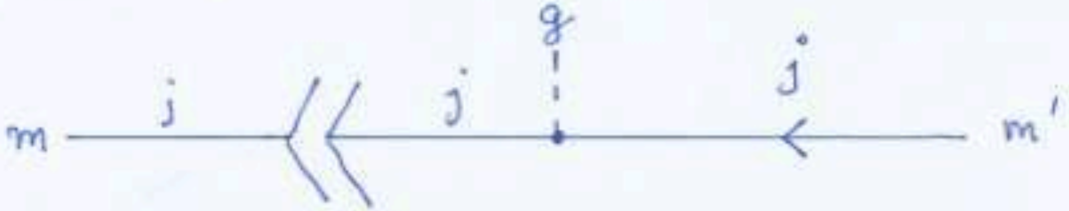


$\langle jm|$ $U(g)$ $|jm'\rangle$

$=$



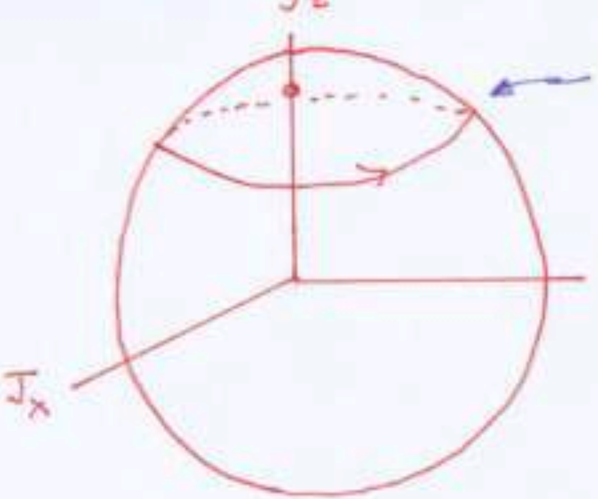
$\langle jm|$ $U(g)|jm'\rangle$ $= \langle B|A\rangle$

One-j Model of D-matrices

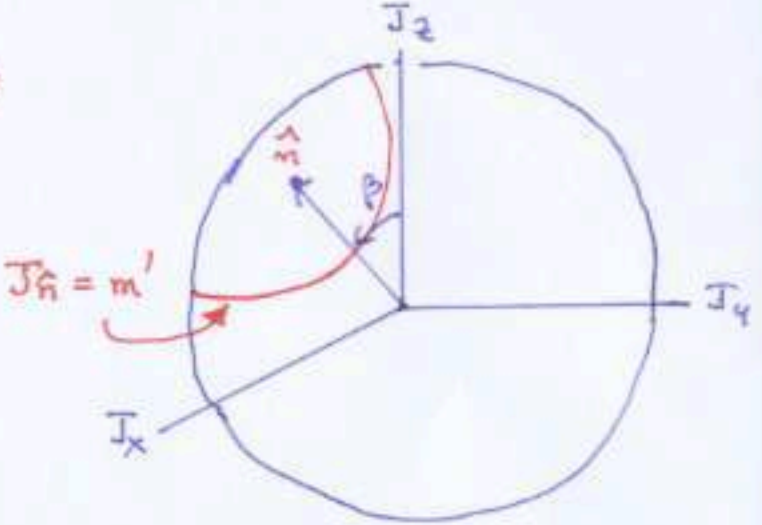
$D^j_{mm'}(g) = \langle B|A\rangle$

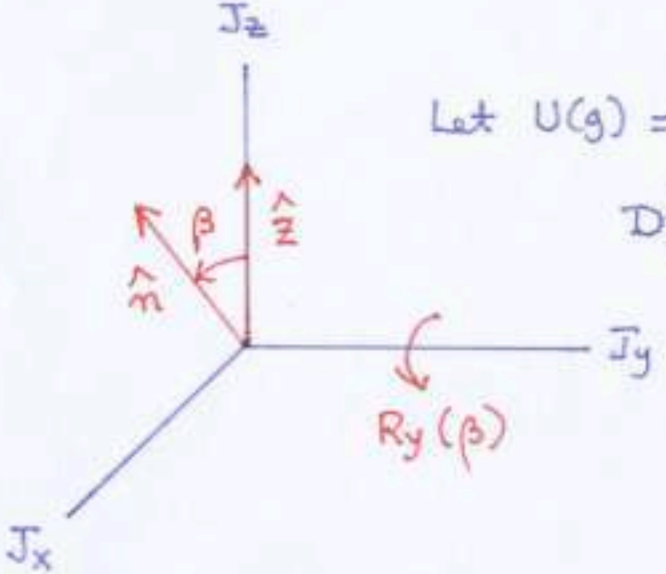
$|B\rangle = |jm\rangle$

$|A\rangle = U(g)|jm'\rangle$



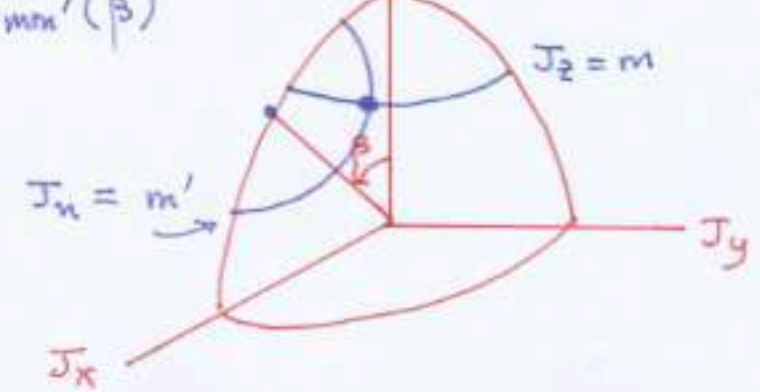
Level set $J_z = m$
(Lagrangian manifold)





Let $U(g) = U_y(\beta)$

$D^j_{mm'}(g) = d^j_{mm'}(\beta)$



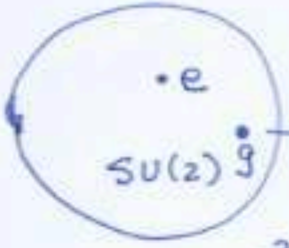
Stationary phase points are intersections of 2 Lagrangian manifolds.

But $D^j_{mm'}(g)$ also = Wave Function on SU(2)

$$D^j_{mm'}(g) = D^j_{mm'}(\alpha\beta\gamma) = \langle \alpha\beta\gamma | j m m' \rangle = \psi_{jmm'}(\alpha\beta\gamma)$$

New Hilbert space, $L^2(SU(2))$.

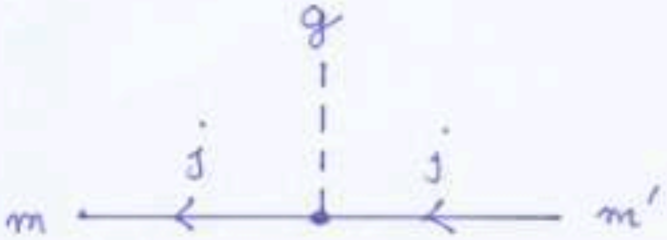
E.g., eigenfunctions of symmetric top.



$\rightarrow g = (\alpha, \beta, \gamma)$

$SU(2) = S^3$

In spin network language:



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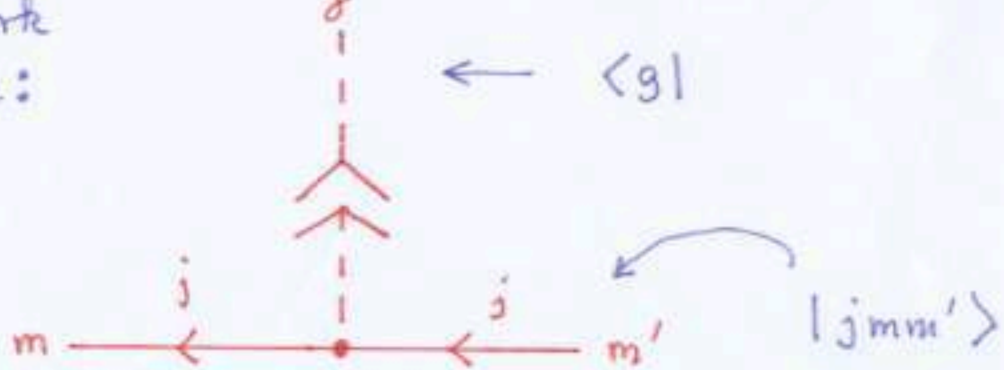


$= D^j_{mm'}(g)$

$= \langle g | j m m' \rangle$

So, semiclassically, must be able to express $D^j_{mm'}(g)$ in terms of Lagrangian manifolds in $T^*(SU(2))$.

Second Way: $U(g): \mathcal{S} \to \mathcal{S}$

QM

$$U(g): \mathcal{S} \to \mathcal{S} = \{\psi(x_1, x_2)\}$$

In Euler angles,

$$U(g) = e^{-i\alpha J_z} \, e^{-i\beta J_y} \, e^{-i\gamma J_z}$$

where

$$\vec{J} = \frac{1}{2} \sum_{\mu\nu} \hat{a}^{+}_{\mu} \, (\vec{\sigma})_{\mu\nu} \, a_{\nu}$$

(Schwinger)

CM

Classically, expect linear symplectic map,

$$: \Phi \longrightarrow \Phi$$

(Remind $\Phi = \mathbb{R}^4 = \mathbb{C}^2$)

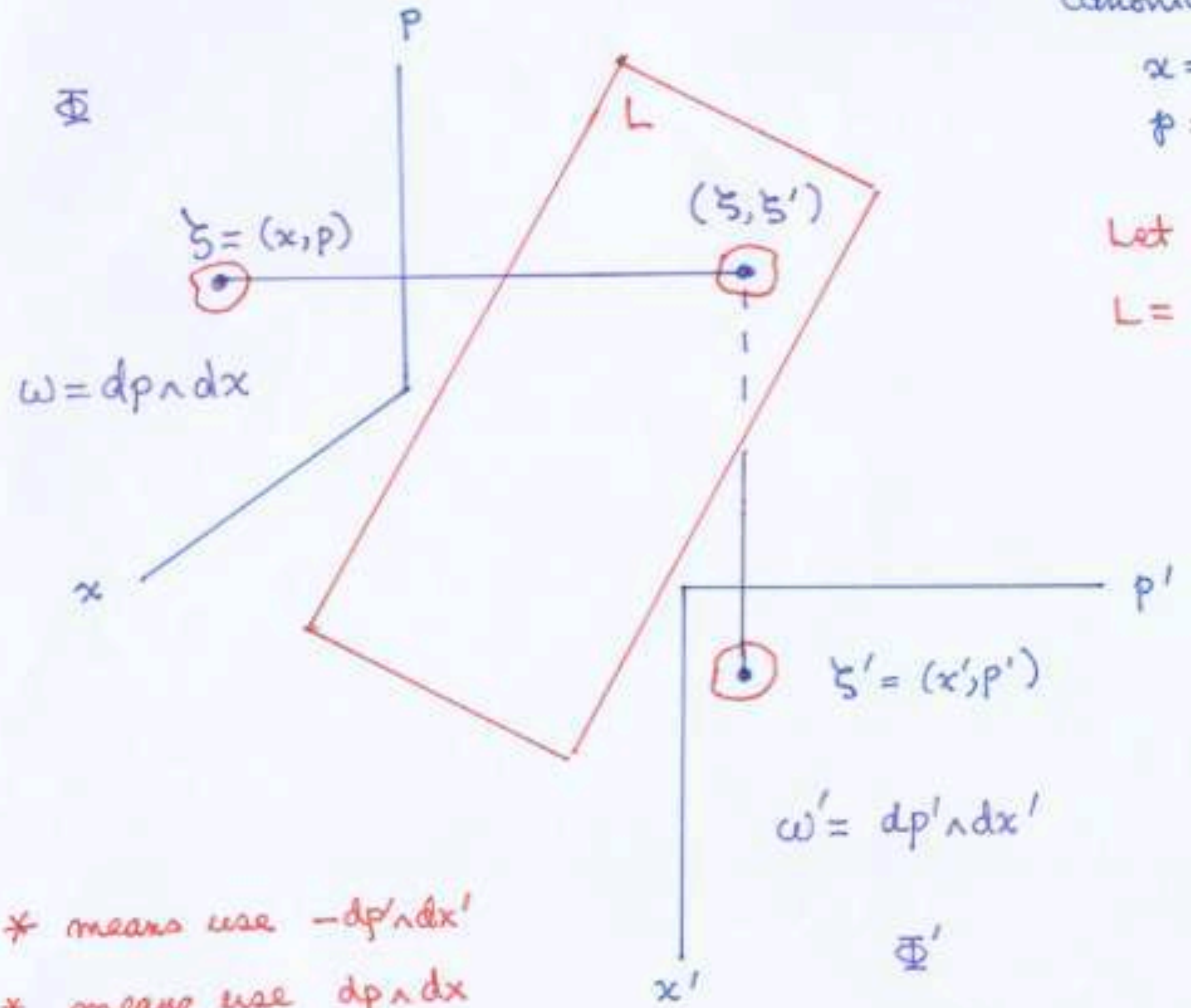
In fact, corresponding to $U(g)$ we have

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto g \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}$$

$$g \in SU(2).$$

A linear, canonical transformation on $\Phi = \mathbb{R}^4 = \mathbb{C}^2$.

Geometry of Canonical Transformations



Canonical Transformation,

$$\left.\begin{aligned} x &= x(x', p') \\ p &= p(x', p') \end{aligned}\right\}.$$

in $\Phi \times \Phi'$

Let L = graph of this C.T., i.e.

$$L = \left\{ (x, p, x', p') \;\middle|\; \begin{aligned} x &= x(x', p') \\ p &= p(x', p') \end{aligned} \right\}.$$

Area preserved:

$$\Omega = \omega - \omega'$$

$$\Omega|_L = 0$$

$$\boxed{\text{L is Lagrangian in } \Phi \times \Phi^*}$$

* means use $-dp' \wedge dx'$

no * means use $dp \wedge dx$

The Doubled Hilbert Space

An operator $\hat{A} : \mathcal{S} \to \mathcal{S}$ can be thought of as living in a doubled Hilbert space. Operator $\hat{A}$ has a doubled “wave function” $\langle x|\hat{A}|x'\rangle$. The “scalar product” of two operators $\hat{A}$ and $\hat{B}$ is

$$\mathrm{tr}(\hat{A}^\dagger \hat{B}) = \int dx\, dx'\, \langle x|\hat{A}|x'\rangle^* \langle x|\hat{B}|x'\rangle.$$

That is,

$$\hat{A} \in \mathcal{H} \otimes \mathcal{H}^*$$

instead of

$$\hat{A} : \mathcal{H} \to \mathcal{H}.$$

For example,

$$D^j_{mm'}(g) = \langle jm|U(g)|jm'\rangle = \mathrm{tr}[(|jm\rangle\langle jm'|)^\dagger U(g)].$$

The Doubled Phase Space

Let's write $D^j_{mm'} = \mathrm{tr}(\hat{A}^\dagger \hat{B})$, where

$$\hat{A} = |jm\rangle\langle jm'|, \qquad \hat{B} = U(g).$$

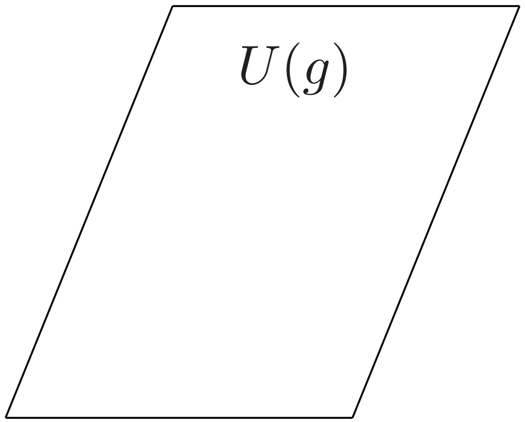
Each of these operators lives in the doubled Hilbert space $\mathcal{H} \otimes \mathcal{H}^*$, so they must be supported by Lagrangian manifolds in the doubled phase space,

$$\Phi \times \Phi^* = \mathbb{C}^2 \times \mathbb{C}^2 = \mathbb{R}^8$$

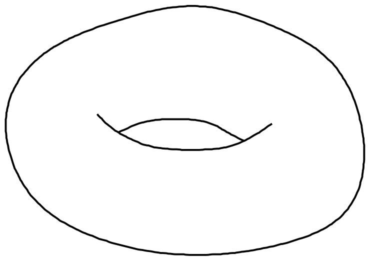
This phase space has real coordinates $(x_1, x_2, p_1, p_2, x'_1, x'_2, p'_1, p'_2)$, or complex coordinates (z_1, z_2, z'_1, z'_2), which can be seen as a pair of "spinors" z and z'.

Lagrangian Manifolds in Doubled Phase Space

As for $U(g)$, the Lagrangian manifold is the graph of the linear canonical transformation, $z = gz'$ (previously plotted). As for $|jm\rangle\langle jm'|$, it is the eigenoperator given by $\hat{I} = j$, $\hat{J}_z = m$, $\hat{I}' = j$, $\hat{J}'_z = m'$.



$z = gz'$



$|jm\rangle\langle jm'|$

$I = j, \qquad J_z = m$

$I' = j, \qquad J_z = m'$