Overview of Results on Spin Networks

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Collaborations with:
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An ideal situation for fruitful interaction:
many analytic results obtained
— waiting to be applied

Outline

Coupling

Fundamental choice for three and four angular momenta: basis of $\mathcal{H}_{j_1} \otimes \mathcal{H}_{j_2} \otimes \mathcal{H}_{j_3} \otimes \mathcal{H}_{j_4}$,

$$\begin{aligned} &1) \quad \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2 = \hat{\mathbf{J}}_{12}, \qquad \hat{\mathbf{J}}_{12} + \hat{\mathbf{J}}_3 + \hat{\mathbf{J}}_4 = 0, \\ &2) \quad \hat{\mathbf{J}}_2 + \hat{\mathbf{J}}_3 = \hat{\mathbf{J}}_{23}, \qquad \hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_{23} + \hat{\mathbf{J}}_4 = 0... \end{aligned}$$

With eigenstates, $|j_1\ j_2\ j_3\ j_4\ j_{12}\ 0\rangle \equiv |j_{12}\rangle$ or $|j_{23}\rangle$.

The $6j$-symbol is the matrix that connects these two bases,

$$\langle j_{12}|j_{23}\rangle = \begin{Bmatrix} j_1 & j_2 & j_{12} \\ j_3 & j_4 & j_{23} \end{Bmatrix}.$$

Democratic Coupling

Alternative: use $\hat{K} = \hat{\mathbf{J}}_1 \cdot (\hat{\mathbf{J}}_2 \times \hat{\mathbf{J}}_3)$ and $\hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2 + \hat{\mathbf{J}}_3 + \hat{\mathbf{J}}_4 = 0$.
Eigenstates,

$$\hat{\mathbf{J}}_1 \cdot (\hat{\mathbf{J}}_2 \times \hat{\mathbf{J}}_3)|k\rangle = k|k\rangle.$$

New kind of, democratic, recoupling coefficient, $\langle j_{12}|k\rangle$.

$$\langle j_{12}|k\rangle = \begin{vmatrix} j_1 & j_2 & j_3 & j_4 \\ & j_{12} & k & \end{vmatrix},$$

this treats all the angular momenta $j_1, \ldots, j_4$ on an equal footing.

Democratic Coupling II

Simple formula for $\hat{K}$,

$$[(\hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2)^2, (\hat{\mathbf{J}}_2 + \hat{\mathbf{J}}_3)^2] = -4i\hat{K}.$$

Leads to matrix elements in terms of $6j$-symbols,

$$\langle j'_{12}|[(\hat{\mathbf{J}}_1 + \hat{\mathbf{J}}_2)^2, (\hat{\mathbf{J}}_2 + \hat{\mathbf{J}}_3)^2]|j_{12}\rangle = \frac{i}{4}(j'_{12}(j'_{12}+1) - j_{12}(j_{12}+1))\langle j'_{12}|(\hat{\mathbf{J}}_2 + \hat{\mathbf{J}}_3)^2|j_{12}\rangle$$

Democratic Coupling III

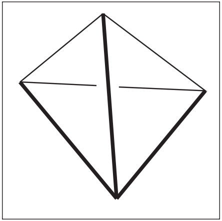
Matrix elements for $\hat{K}$ (Jacobi Matrix),

$$K = \begin{pmatrix} 0 & -i\alpha_1 & 0 & \dots & \dots \\ i\alpha_1 & 0 & -i\alpha_2 & \ddots & \\ 0 & i\alpha_2 & 0 & -i\alpha_3 & \\ \vdots & \ddots & i\alpha_3 & \ddots & \ddots \\ & & & \ddots & 0 \end{pmatrix},$$

$\alpha_i = \alpha_i(\{6j\})$.

Overview

Strategy: Bohr-Sommerfeld Quantization



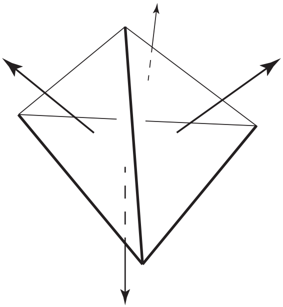
A tetrahedral grain of space

Need: A classical dynamical system,

- kinematics (phase space and Poisson brackets $\{f, g\}$)
- dynamics H.

The area vectors of a tetrahedron determine its shape:

$$\vec{A}_1 + \vec{A}_2 + \vec{A}_3 + \vec{A}_4 = 0.$$



Kinematics: Penrose

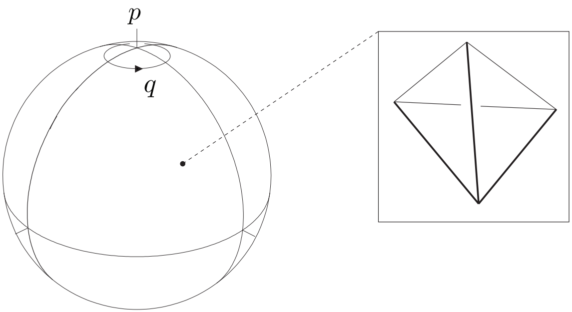
- Physical input: $\vec{A}_1, \ldots, \vec{A}_4$ are angular momenta

Angular momenta have Poisson brackets,

$$\{f, g\} = \sum_{l=1}^{4} \vec{A}_l \cdot \left(\frac{\partial f}{\partial \vec{A}_l} \times \frac{\partial g}{\partial \vec{A}_l} \right).$$

Kinematics II: Kapovich & Millson

$\vec{A}_1, \ldots, \vec{A}_4$ angular momenta



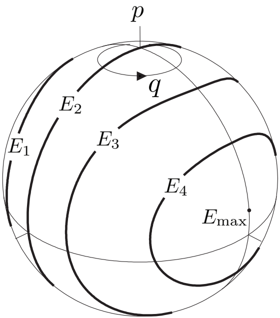
$$p = |\vec{A}_1 + \vec{A}_2| \qquad q = \text{Angle of rotation generated by } p\text{:}$$

$$\{q, p\} = 1$$

Dynamics

Take as Hamiltonian the volume:

$$H = V = \sqrt{|V^2|} = \frac{\sqrt{2}}{3}\sqrt{|\vec{A}_1 \cdot (\vec{A}_2 \times \vec{A}_3)|}.$$



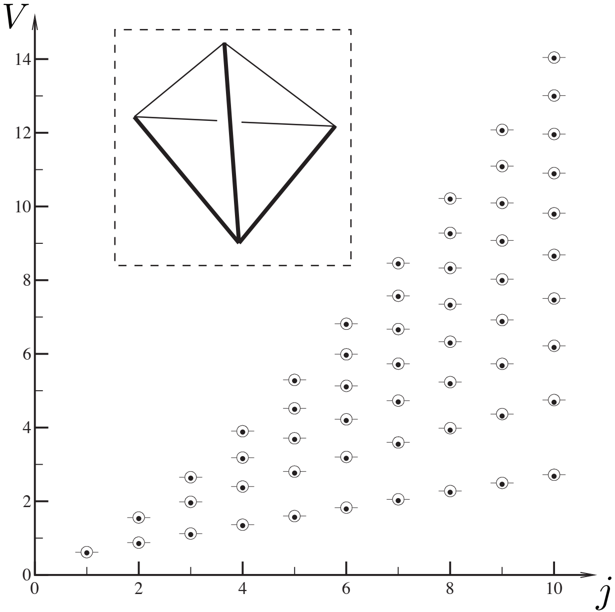
Bohr-Sommerfeld Quantization

Area of orbits given in terms of elliptic functions

$$S(E) = \left(\sum_{i=1}^{4} a_i \, K(m) - \sum_{i=1}^{4} b_i \, \Pi(\alpha_i^2, m) \right) E.$$

Require Bohr-Sommerfeld quantization condition,

$$S = (n + 1/2) 2\pi\hbar.$$



$A_1 = j + 1/2$

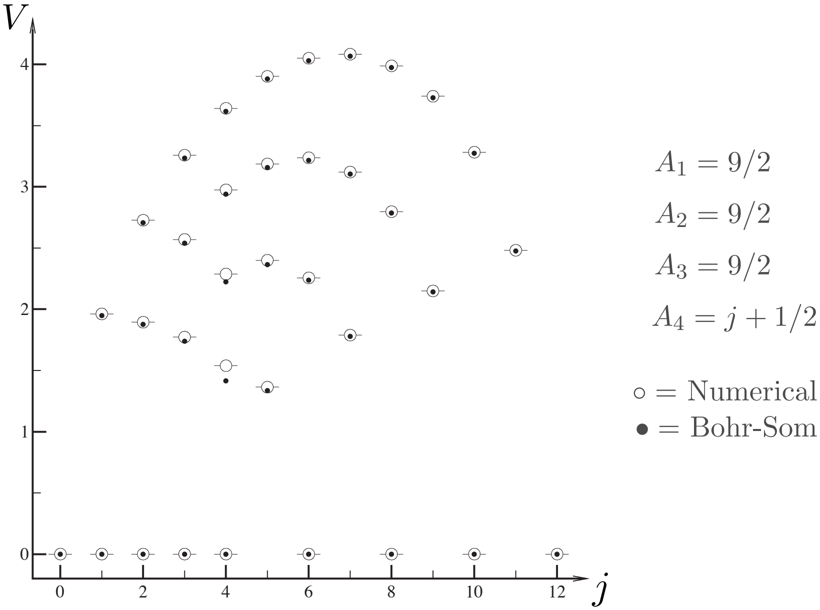
$A_2 = j + 1/2$

$A_3 = j + 1/2$

$A_4 = j + 3/2$

○ = Numerical

● = Bohr-Som



V
4
3
2
1
0
0
2
4
6
8
10
12
j
$A_1 = 9/2$
$A_2 = 9/2$
$A_3 = 9/2$
$A_4 = j + 1/2$
○ = Numerical
● = Bohr-Som

Table: Volume spectrom			
$j_1 j_2 j_3 j_4$	Numerical	Bohr-Sommerfeld	Accuracy
$\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}$	0.3102	0.2523	0.19
$1\ 1\ \frac{1}{2}\ \frac{1}{2}$	0.3964	0.3440	0.13
$\frac{3}{2}\ \frac{3}{2}\ \frac{1}{2}\ \frac{1}{2}$	0.4638	0.4061	0.12
$\frac{3}{2}\ 1\ 1\ \frac{1}{2}$	0.4984	0.4584	0.08
$1\ 1\ 1\ 1$	0 0.6204	0 0.5658	0 0.09
$2\ 2\ \frac{1}{2}\ \frac{1}{2}$	0.5216	0.4581	0.12
$2\ \frac{3}{2}\ 1\ \frac{1}{2}$	0.5773	0.5354	0.07
$2\ 1\ 1\ 1$	0.6204	0.5975	0.04
$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{1}{2}$	0.6204	0.5975	0.04

Table: Volume spectrom

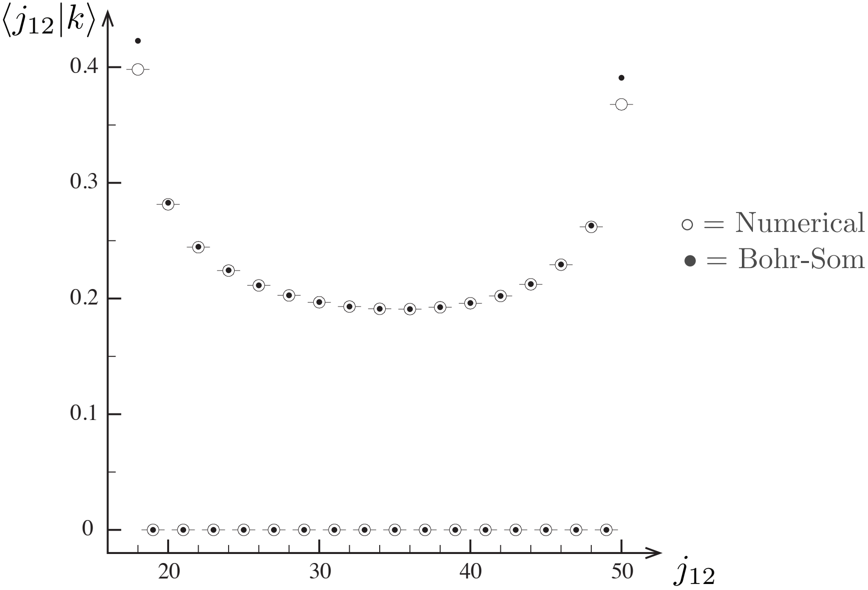
$j_1 j_2 j_3 j_4$	Numerical	Bohr-Sommerfeld	Accuracy
...			
$2\ \frac{3}{2}\ \frac{3}{2}\ 1$	0 0.9036	0 0.8676	0 0.04
$\frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}\ \frac{3}{2}$	0.5372 0.9929	0.4521 0.9473	0.16 0.05
...			
6 6 6 7	1.8276 3.2039 4.2249 5.1328 5.9891 6.8173	1.7949 3.1618 4.1895 5.1053 5.9673 6.7994	0.018 0.013 0.008 0.005 0.004 0.003

Democratic Recoupling

WKB quantization can also be used to find $\langle j_{12}|k\rangle$, thought of as a wavefunction,

$$\psi_k(j_{12}) = \langle j_{12}|k\rangle = A\cos\left(S - \frac{\pi}{4}\right),$$

with S expressed in terms of incomplete elliptic integrals of the first and third kinds.


$\langle j_{12}|k\rangle$
0.4
0.3
0.2
0.1
0
20
30
40
50
j_{12}
○ = Numerical
● = Bohr-Som

Scaling I

Previous result on scaling (Brunnemann and Thiemann):

$$v_{\max} \sim j_{\max}^{3/2},$$

with $j_{\max}$ the largest of the four spins.

Using Bohr-Sommerfeld:
Equal areas $A = j + 1/2$,

$$v_{\max} = \frac{2^{3/2}}{3^{7/4}} A^{3/2} \sqrt{1 - \frac{3}{4A}}$$

Scaling II

Previous result on scaling (Brunnemann and Thiemann):

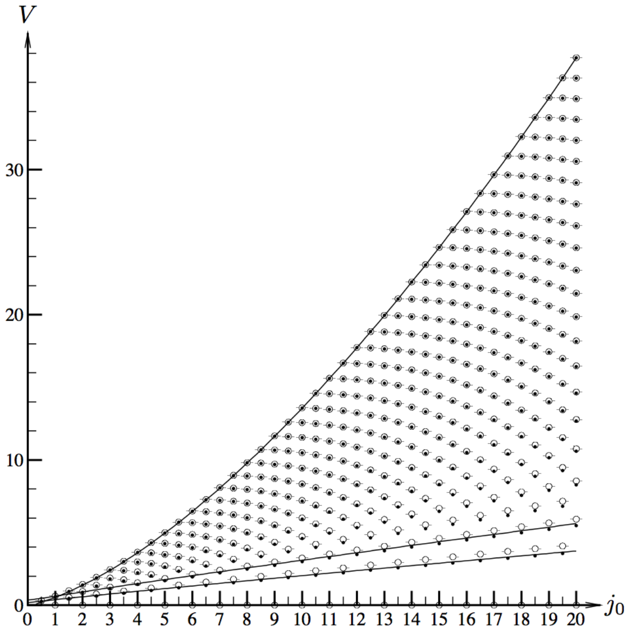
$$v_{\text{min}} \sim j_{\text{max}}^{1/2}.$$

New results on scaling:
For small elliptic parameter ($m << 1$),

$$v_{\text{min}} \sim c(A_1 A_2 A_3 A_4)^{1/4},$$

$c = 2/3$ and $c = \sqrt{2}/3$ if $\dim \mathcal{H}$ is odd or even respectively.
Equal areas A:

$$v_{\text{min}} = \frac{2\sqrt{\pi}}{3\sqrt{3}} A \left(\frac{1}{\sqrt{\ln\left(\frac{6eA}{\pi}\right) + \ln\left(\ln\left(\frac{6eA}{\pi}\right)\right) + \cdots}} \right)$$



V
30
20
10
0
0 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20
j_0

Outline

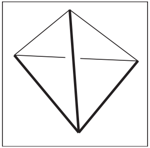
Ponzano-Regge Model

The $6j$-symbol has an incredible asymptotic limit,

$$\begin{Bmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{Bmatrix} \sim \frac{1}{\sqrt{12V}} \cos\left(\sum_r J_r\theta_r + \frac{\pi}{4}\right)$$

Here

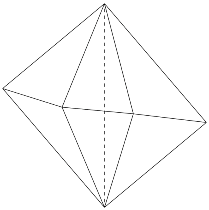
- V is the volume of the tetrahedron with edges $J_r \equiv j_r + \frac{1}{2}$.
- θ_r are the dihedral angles between the faces.



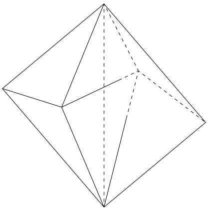
9j-Symbol

The 9j-Symbol is the matrix element that transforms between LS- and jj- coupling bases:

$$\langle SL | j_1 j_2 \rangle \sim \begin{Bmatrix} s_1 & s_2 & S \\ l_1 & l_2 & L \\ j_1 & j_2 & J \end{Bmatrix}$$



First configuration

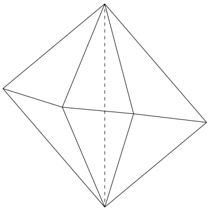


Second configuration

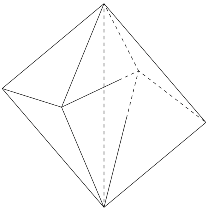
9j-Symbol

The 9j-Symbol is the matrix element that transforms between LS- and jj- coupling bases:

$$\langle SL | j_1 j_2 \rangle \sim \begin{Bmatrix} s_1 & s_2 & S \\ l_1 & l_2 & L \\ j_1 & j_2 & J \end{Bmatrix} = \begin{Bmatrix} 129/2 & 137/2 & S \\ 113/2 & 121/2 & L \\ 64 & 108 & 90 \end{Bmatrix}$$



First configuration



Second configuration

The Asymptotic Formula

The asymptotic formula for the $9j$-symbol is,

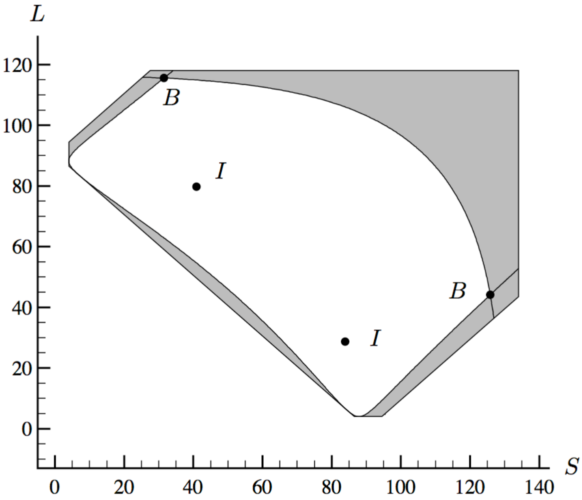
$$\begin{Bmatrix} s_1 & s_2 & S \\ l_1 & l_2 & L \\ j_1 & j_2 & J \end{Bmatrix} = A_1 \cos\left(\sum_{i=1}^{9} J_i\theta_i\right)_1 + A_2 \sin\left(\sum_{i=1}^{9} J_i\theta_i\right)_2 ,$$

where,

$$A_{1,2} = \frac{1}{4\pi\sqrt{|\det M|}}\bigg|_{1,2} \qquad \text{and} \qquad M \sim \begin{pmatrix} \{|\vec{j_1}|, |\vec{S}|\} & \{|\vec{j_1}|, |\vec{L}|\} \\ \{|\vec{j_2}|, |\vec{S}|\} & \{|\vec{j_2}|, |\vec{L}|\} \end{pmatrix}$$

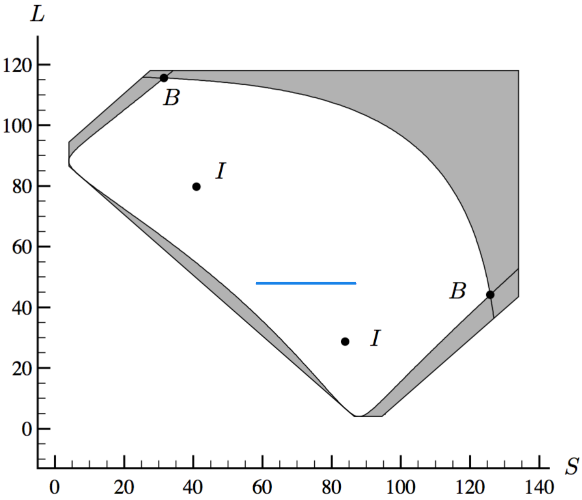
$$A_{1,2} = \frac{1}{4\pi\sqrt{|V_a V_b - V_c V_d|_{1,2}}}$$

Evidence



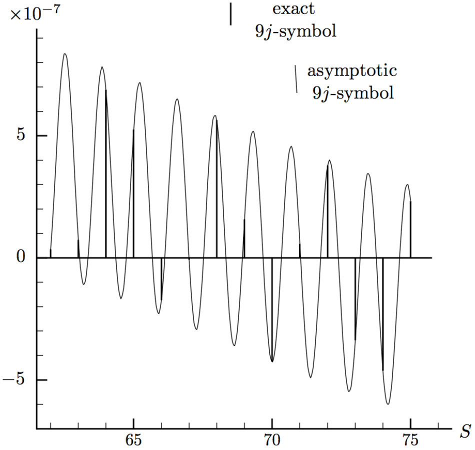
Classically allowed and forbidden regions

Evidence



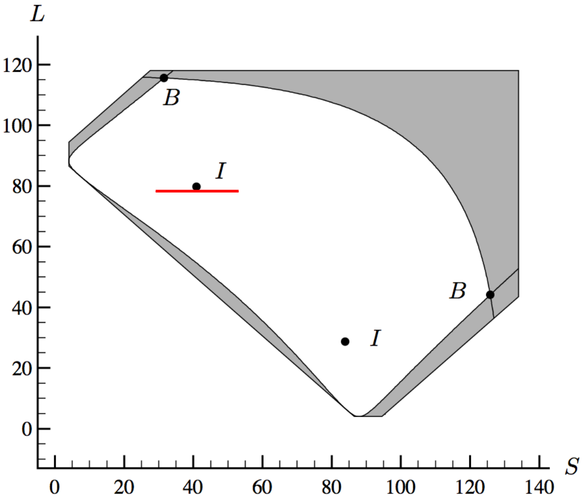
Classically allowed and forbidden regions

Evidence



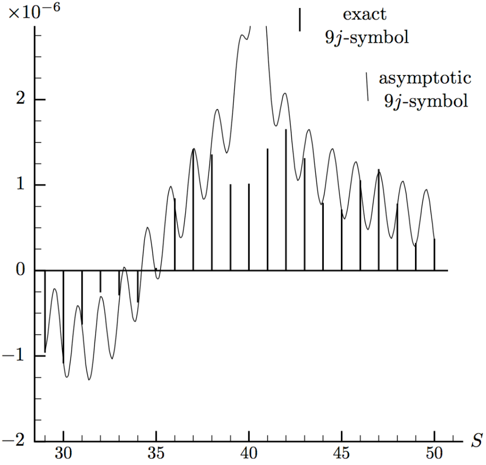
Comparison of exact and asymptotic values of the $9j$-symbol

Evidence

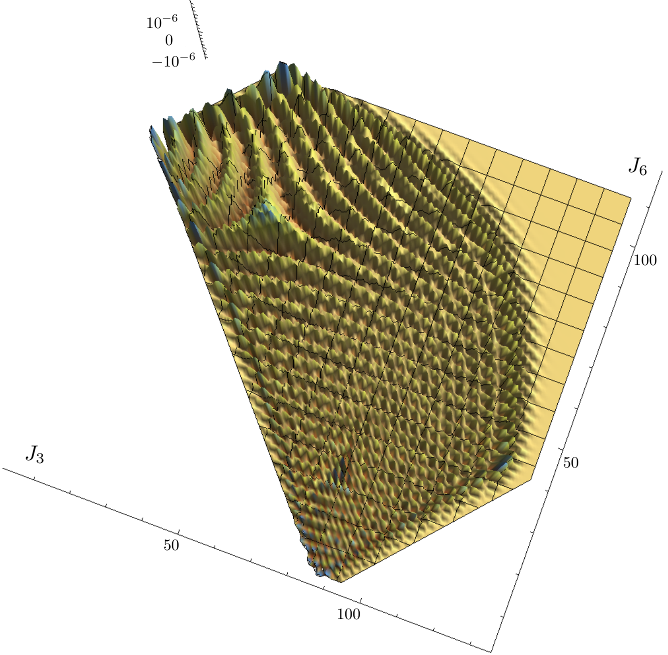


Classically allowed and forbidden regions

Evidence



Exact and asymptotic values of the $9j$-symbol near a caustic



10^{-6}
0
-10^{-6}
J_6
100
50
J_3
50
100

Outline

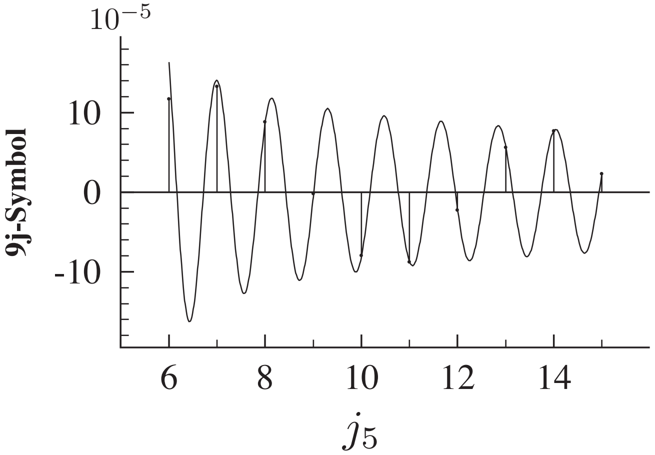
Large and Small Angular Momenta

Work by Robert Littlejohn and Liang Yu.

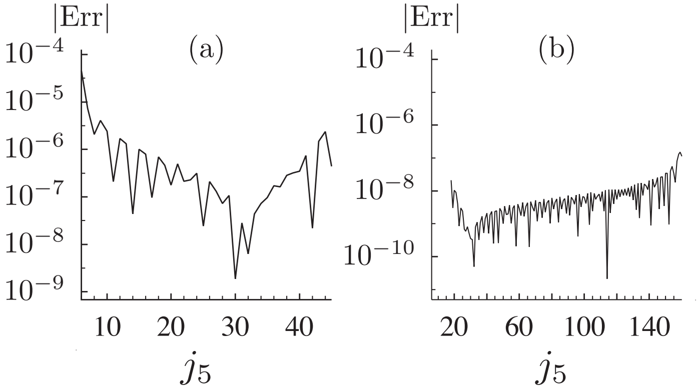
- Idea: Treat large angular momenta semiclassically and small angular momenta by exact linear algebra

$$\begin{Bmatrix} j_1 & j_2 & j_{12} \\ s & j_4 & j_34 \\ j_{13} & j_{24} & j_5 \end{Bmatrix} = \begin{Bmatrix} 51/2 & 53/2 & 28 \\ 1/2 & 47/2 & 24 \\ 25 & 27 & j_5 \end{Bmatrix}$$

Evidence



Exact and asymptotic values of the $9j$-symbol with small s



Comparison of the Error for the given js (a) and after they are doubled (b)

Conclusions & Acknowledgements

- Remarkably simple road to the quantization of geometry.
- Do these results have equally nice applications to three body dynamics in chemistry?
- The $9j$-symbol has a beautifully simple asymptotics with very interesting geometry behind it. How can this be used to deepen our understanding of why processes involving the symbol proceed as they do?

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