I Last time

II Clean up

III Gravitational Time Dilation as Geometry

IV Particle Motion in Curved Spacetime

- A more careful argument using the equivalence principle and a rocket gave;

\[ \Delta \tau_B = \frac{\Delta \tau_A}{\left(1 + \frac{gh}{c^2}\right)} \]

- The slogan I use is "Sloshing clocks run slow."

- The acceleration g refers to the acceleration due to gravity at the surface of the Earth. We'd like to

We treated gravitational time dilation twice.

- Naively you can drop a photon down a mine shaft!

\[ \omega_2 = \omega_1 \left(1 + \frac{gh}{c^2}\right) \]

Useful for reconstructing the result quickly.

have a more general expression. We'll use the gravitational potential \( \Phi \). (N.B. this is defined as the potential energy per unit mass, i.e. \( \Phi = \frac{-GM}{r} \) units of \( \frac{\text{Energy}}{\text{mass}} \). Not \( U = \frac{-GMm}{r} \) and \( [\Phi] = \frac{\text{Energy}}{\text{mass}} \).

So, \( gh = \Phi_A - \Phi_B \).

If we also use the reception rates, \( \tau_A = \frac{1}{\Delta t_A} \), \( \tau_B = \frac{1}{\Delta t_B} \).
then,
\[
\begin{align*}
\text{rate}_B &= \text{rate}_A \left(1 + \frac{g_0^B}{c^2}\right) \\
&= \text{rate}_A \left(1 + \frac{\overline{\Phi}_A - \overline{\Phi}_B}{c^2}\right).
\end{align*}
\]

III. We begin our program in earnest: I hand you a metric, say a little about it and then you explore it.

Our first curved spacetime metric:
\[
\begin{align*}
\text{d}s^2 &= -(1 + \frac{2\overline{\Phi}(x^i)}{c^2})(\text{d}t)^2 + (1 - \frac{2\overline{\Phi}(x^i)}{c^2})(\text{d}x^2 + \text{d}y^2 + \text{d}z^2) \\
&= \text{"Static weak field metric."}
\end{align*}
\]

What does light do in this curved geometry? Light rays are not straight lines. But, the metric is static, so the shape of two light rays at different times is the same.

Important calculation: What are \(\Delta \tau_A\) and \(\Delta \tau_B\) to first order in \(\frac{\overline{\Phi}(x^i)}{c^2}\)?

Well,
\[
\Delta \tau_A^2 = -\frac{\Delta s^2}{c^2} = (1 + \frac{2\overline{\Phi}_A}{c^2}) \frac{c^2 \Delta t^2}{c^2},
\]

where \(\overline{\Phi}_A \equiv \overline{\Phi}(x_A)\). Then
\[
\Delta \tau_A = \sqrt{1 + \frac{2\overline{\Phi}_A}{c^2}} \Delta t \approx \left(1 + \frac{\overline{\Phi}_A}{c^2}\right) \Delta t.
\]

Very similarly, \(\Delta \tau_B \approx (1 + \frac{\overline{\Phi}_B}{c^2}) \Delta t\)

\[
\Rightarrow \Delta \tau_B = \left(1 + \frac{\overline{\Phi}_B}{c^2}\right) \Delta \tau_A
\]

Here \(\overline{\Phi}_i = \overline{\Phi}(x^i)\) is shorthand for \(\overline{\Phi}(x,y,z)\) is \(\Phi(\zeta)\) the gravitational potential in the Newtonian sense just discussed and is a function of position alone.
But then,

\[ \Delta \tau_B \approx (1 + \frac{\Delta \tau_B}{c^2})(1 - \frac{\Delta \tau_A}{c^2}) \Delta \tau_A \]

\[ = (1 + \frac{\Delta \tau_B - \Delta \tau_A}{c^2}) \Delta \tau_A + \ldots \]

Same result as before! What part of the line element mattered most for this calculation? It was the curvature in time that mattered — the spatial metric played no role!

room for a new answer:

**Variational Principle**

for **Free Particle Motion**

The world line of a free particle between two timelike separated pts extremizes the proper time between them.

Let's try it: **Example**: Flat Minkowski spacetime

\[ \tau_{AB} = \int_{A}^{B} \sqrt{-ds^2} = \int_{A}^{B} ds \]

**IV How do particles move in a curved spacetime?**

**Reminder**: In mechanics we have

\[ S = \int_{t_A}^{t_B} L dt, \quad L = T - V \]

and the eq. of motion follow from

\[ SS = 0 \iff \frac{dL}{dt} \equiv \frac{1}{dt} (\frac{\partial L}{\partial \dot{x}}) = 0 \]

This was an alternative axiomatization to Newton's \( \vec{F} = m \vec{a} \).

Our geometrical context makes

\[ s = \int_{A}^{B} (dt^2 - (dx^2 + dy^2 + dz^2))^{1/2} \]