Lecture 8
Sept. 14th, 2011

Example: Plane Polar Coords

Last time we started to find the Euler-Lagrange eq's for a particle moving in the plane and described with plane polar coords.

Why? Plane polar coords are an example of generalized coords. If our system had

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{r}} \right) = \ddot{r} = \dot{\theta} \frac{d}{dt}(mr) = m \ddot{r} \]

Now, \(-\partial U/\partial r = F_r\) and so,

\[ F_r = m \ddot{r} - m \dot{\theta} \dot{\phi} = m (\ddot{r} - r \dot{\phi}^2) = m a_r \]

So simple compared to the usual derivation of \(a_r\)!

Let's review it this derivation:

"Circular" symmetry they would be a good choice. We found

\[ T = \frac{m}{2}(\dot{r}^2 + r^2 \dot{\phi}^2) \]

The potential in these coords is

\[ U = U(r, \phi) \]

So,

\[ L = \frac{1}{2} m (\dot{r}^2 + r^2 \dot{\phi}^2) - U(r, \phi) \]

and we can calculate:

\[ \frac{\partial L}{\partial r} = m \dot{\phi}^2 - \frac{\partial U}{\partial r} \]
Write \( \hat{r} = r \hat{\phi} \) so that
\[
\hat{r} = \cos \phi \hat{x} + \sin \phi \hat{y}
\]
and
\[
\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}
\]
Want to find \( a_r = \hat{r} \)-component
of \( \ddot{a} = \ddot{r} \). So, calculate
\[
\ddot{r} = \dddot{r} \hat{r} + \ddot{r} \hat{\phi}
\]
and we need \( \ddot{r} \). Well
\[
\ddot{r} = -\dot{\phi} \phi \hat{x} + \dot{\phi} ^2 \hat{y}
\]
\[
= \dot{\phi} (-\dot{\phi} \hat{x} + \ddot{\phi} \hat{y}) = \dot{\phi} \hat{\phi}
\]
The Euler-La Grange Equations are an amazing computational gift.

Let's do the \( \phi \) component too:
\[
\frac{d}{dt} \left( \frac{\partial L}{\partial \ddot{\phi}} \right) = \frac{\partial}{\partial \phi} \left( \frac{\partial L}{\partial \dot{\phi}} \right)
\]
\[
\Rightarrow \frac{\partial}{\partial \phi} \left( -m \dot{\phi} \ddot{\phi} \right) = \dot{\phi} \hat{\phi}
\]
Thus in turn implies call it
\[-m \ddot{\phi} = r F_\phi = \text{torque} = \Gamma
\]
Meanwhile, moment of inertia
\[m r^2 \phi = \mathbf{I} \omega = \mathbf{L} \]
angular velocity \( \omega \) angular momentum
So, the \( \phi \) E-L equation says that
\[
\Gamma = \frac{1}{\dot{\rho}}
\]
torque = time derivative of angular momentum!

The E-L equation automatically "knew" that \( \phi \) was an angular coordinate.

III.

This observation leads us to make two reasonable definitions. The E-L. eq's for generalized coords \( \xi_i \) \( (i = 1, \ldots, n) \)
are
\[
\frac{\partial I}{\partial \dot{\xi}_i} = \frac{d}{dt} \left( \frac{\partial I}{\partial \dot{\xi}_i} \right) \quad (i = 1, \ldots, n)
\]

Last time we saw that the E-L. equations took the same form in every coordinate system. Here we've also seen
that they're very efficient. One can't help but ask: why? (!) This is a broad question but one answer is
that \( I \) is a scalar. This is what allowed us to change coordinates and still end up with
\[ S = \int I \, dt \] and the E-L. equations.

We define
\[
\frac{\partial I}{\partial \dot{\xi}_i} = \left( \text{ith component of a generalized force} \right)
\]
for example
\[
\frac{\partial I}{\partial \dot{\xi}_i} = \left( \text{a force, or a torque} \right)
\]
These are the or...

most common cases

\[
\frac{\partial I}{\partial \dot{\xi}_i} = \left( \text{ith component of a generalized momentum} \right)
\]
for example
\[
\frac{\partial I}{\partial \dot{\xi}_i} = \left( \text{a momentum or an angular momentum or ...} \right)
\]

In modern physics we exploit this fact to the hilt: we write down every known scalar consistent with the symmetries of the system and throw them into \( I \). This has become a central principle for constructing physical theories. More on symmetry next week.
The bead is constrained to move along the wire. In particular if you know $\theta(t)$ then you can find $x(t)$, $y(t)$ and $z(t)$ (given, say, that the hoop is in the $xz$-plane at $t=0$). Try it. We say that the bead has one degree of freedom.

Let's find the Lagrangian. We choose an inertial frame, say the

Gravitational potential energy is

$$U = mg \left( R - R \cos \theta \right)$$

$$= mgR \left( 1 - \cos \theta \right)$$

So

$$L = \frac{1}{2} m \left( \dot{R}^2 + \dot{R}^2 \dot{\theta}^2 + R^2 \sin^2 \theta \omega^2 \right)$$

$$- mgR \left( 1 - \cos \theta \right)$$

and E.-H. e.g. is

$$\frac{2x}{\dot{\theta}} = \frac{m R^2 \sin \theta \cos \theta \dot{\omega}^2 - mgR \sin \theta}{\dot{\theta}}$$

$$\frac{1}{t} \left( \frac{2x}{\dot{\theta}} \right) = \frac{d}{dt} \left( mR^2 \dot{\theta} \right) = mR^2 \dot{\theta}$$
$$mR^2 \left( 6 \cos \omega^2 - \frac{g}{R} \sin \theta \right) = mR^2 \ddot{\theta}$$

$$\ddot{\theta} = (6 \cos \omega^2 - \frac{g}{R}) \ddot{\theta}$$

But wait, was I justified in applying the E.-L. eq.? We've only shown that it works for unconstrained systems. You should check this equation with Newton's equations.

N particles in 3D = 3N D.O.F.

When the # of D.O.F. of N particles in 3D is less than 3N, we say system is constrained.

Next time: We'll show whether the E.-L. equations apply to constrained systems.

Setup: Definition of degrees of freedom in general:

# of D.O.F. = # of coords that can be independently varied in a small displacement.

E.g. pendulum = 1 D.O.F.
double pendulum = 2 D.O.F.