Today's Outline

I Last Lecture

II The stress Tensor

Lecture 38

Final Lecture

December 2nd, 2011

I Last Lecture

- We explored some examples of surface forces
- We showed that for a non-viscous medium pressure is isotropic
- We introduced stress, strain and elastic moduli:
  \[ \text{stress} = \frac{\text{force}}{\text{area}} \]

II The stress Tensor

Let's derive the general expression for surface force on a small area \( dA \) of a closed surface \( S \) in a continuous medium.

First let \( dA = \alpha dA \). We'll proceed in two steps:

(i) Show that \( \vec{F}(\alpha_1 dA_1 + \alpha_2 dA_2) = \alpha_1 \vec{F}(dA_1) + \alpha_2 \vec{F}(dA_2) \)

strain = fractional deformation

elastic modulus = \( \frac{\text{stress}}{\text{corresponding strain}} \)

Before taking a bird's eye view of the course and its applications we introduce one more object from continuum mechanics, the strain tensor. This gives us one more opportunity to explore a practical tensor.
(ii) From (i) we argue that it follows that \( \vec{F} \) and \( \vec{dA} \) are related by a tensor \( \mathbf{\Sigma} \), the "stress tensor."

Let’s start with (i). By the definition of a surface force

\[ \vec{F}(\alpha \, d\vec{A}) = \alpha \, \vec{F}(d\vec{A}) \]

for \( \alpha \) positive and not too large. Now \( d\vec{A} \rightarrow -d\vec{A} \) switches inside and outside.

It’s not too hard to prove that

\[ d\vec{A}_1 + d\vec{A}_2 + d\vec{A}_3 = 0 \]

in general but here’s a nice intuitive argument. Imagine the prism is immersed in a fluid that is non-viscous then because it’s in equilibrium the surface forces satisfy

\[ \vec{F}_1 + \vec{F}_2 + \vec{F}_3 = 0 \]

and by Newton’s 3rd law \( \frac{d}{d} \)

this also switches \( \vec{F} \) and \( -\vec{F} \) so,

\[ \vec{F}(-d\vec{A}) = -\vec{F}(d\vec{A}) \]

Now, what about adding \( d\vec{A} \)’s?

Consider again a triangular prism

![Diagram of a triangular prism](image)

but they are just pressures so

\[ p \, d\vec{A}_1 + p \, d\vec{A}_2 + p \, d\vec{A}_3 = 0 \]

\[ \Rightarrow d\vec{A}_1 + d\vec{A}_2 + d\vec{A}_3 = 0 \]

This last result is geometrical and can’t depend on how we arrived at, so it’s totally general.

Now, immerse the prism in...
in any medium (or consider a prism shaped piece of any material) then

$$\vec{F}(d\vec{A}_1 + d\vec{A}_2) = \vec{F}(-d\vec{A}_3)$$

$$= -\vec{F}(d\vec{A}_3)$$

$$= \vec{F}(d\vec{A}_1) + \vec{F}(d\vec{A}_2)$$

where the last equality follows from the equilibrium condition on tensor connecting \( \vec{F} \) and \( d\vec{A} \):

$$\vec{F} = \sum \vec{F}_i$$

with

$$[\sum \delta_{ij} = \sigma_{ij}$$

the "stress tensor". If we were to go on from here we would derive a strain tensor and the E.O.M. would the forces. That does it, Part 3.3

the surface force depends linearly on \( d\vec{A} \).

(ii) But linearity means that we can write

$$F_i = \sum \sigma_{ij} dA_j$$

which is precisely what it means to say there is a follow from Newton's laws and the relationship between these tensors. Instead, we will briefly comment on the course as a whole. See attached slides.
The Pioneer anomaly is an unexplained acceleration towards the Sun with magnitude,

\[ 8.74 \pm 1.33 \times 10^{-10} \text{ m/s}^2. \]
The current best explanation...
...is completely classical.

In fact, it uses ray tracing techniques developed in the ’70s and mostly used for video games!
A little quantum chaos...
... and a meditation on quantum gravity.

\(\hbar\) is crazy small \(6.626 \times 10^{-34}\). In fact, we are closer to the size of the observable universe, 45.7 billion light years or \(4 \times 10^{26}\) meters than we are...

... to the Planck length, \(\ell_{Pl} = \sqrt{\frac{\hbar G}{c^3}} = 1.6 \times 10^{-35}\) m.