Today's Outline

I Last Lecture:
Completed discussion of normal modes

II Introduction to Chaos & Nonlinear Mechanics

Lecture 30
November 9th, 2011

Last Lecture Found normal mode E.O.M.

\[ \ddot{\mathbf{\ddot{\mathbf{G}}}} = -\mathbf{K} \mathbf{\ddot{\mathbf{G}}} \]

with

\[ M_{jk} = A_{jk}(0) = \sum_{\alpha} e^{i\omega_{\alpha}} \left( \frac{\partial^{2} \phi_{\alpha}}{\partial \phi_{j} \partial \phi_{k}} \right) \]

evaluated at \( \mathbf{0} \rightarrow \mathbf{\ddot{\mathbf{G}}} \)

and

\[ K_{jk} = \frac{\partial^{2} U}{\partial \phi_{j} \partial \phi_{k}} \bigg|_{\mathbf{\ddot{\mathbf{G}}} = \mathbf{0}} \]

Notice that you, in fact, can't write down the Lagrangian at all for these problems; you can just find \( \mathbf{M} \) and \( \mathbf{K} \) and proceed as before.

Guess: \( \mathbf{\ddot{\mathbf{G}}}(t) = \mathbf{Re} \, \mathbf{\ddot{\mathbf{L}}}(t), \mathbf{\ddot{\mathbf{L}}}(t) = \mathbf{a} e^{i\omega t} \)

and

\[ \det(\mathbf{K} - \omega^{2} \mathbf{M}) = 0 \] Normal frequency

Solve:

\[ (\mathbf{K} - \omega^{2} \mathbf{M}) \mathbf{a} = 0 \] Normal modes

Your text goes through another example:

\[ \text{Example} \]

Check it out!

II Chaos & Nonlinear Mechanics

A system is said to be chaotic if it obeys deterministic E.O.M. but still the motion cannot be predicted on long time scales.
Most realistic systems are nonlinear:

\[ \text{e.g., } m \dddot{x} = -GmM \frac{\ddot{r}}{r^2} \text{ or } \frac{d^2 x}{dt^2} + \mu(x^2-1) \frac{dx}{dt} + x = 0 \]

Often, but not always, nonlinear systems exhibit chaos.

Nonlinearity changes many things but important amongst these is that nonlinear systems do not allow superposition.

\[ D(\alpha_1 x(t) + \alpha_2 x_2(t)) = \alpha_1 Dx_1 + \alpha_2 Dx_2 \quad \text{[linear]} \]

We'll explore all of this with a particular nonlinear system: the damped driven pendulum (DDP):

\[ \begin{align*}
\ddot{\phi} + \frac{b}{m} \dot{\phi} + \frac{g}{L} \sin \phi &= \frac{F_0}{mL} \cos(\omega t) \\
\end{align*} \]

The E.O.M. is:

\[ I \dddot{\phi} = F \]

\[ \Rightarrow mL^2 \dddot{\phi} = -mgL \sin \phi - bL \dot{\phi} + FL(t) \]

\[ = -mgL \sin \phi - bL^2 \dot{\phi} + FL_0 \cos(\omega t) \]

Assume sinusoidal drive at drive frequency \( \omega \).

\[ \Rightarrow \dddot{\phi} + \frac{b}{m} \ddot{\phi} + \frac{g}{L} \sin \phi = \frac{F_0}{mL} \cos(\omega t) \]

Let \( 2p = \frac{b}{m} \), \( p \) the "damping constant"

\[ D(\alpha_1 x(t) + \alpha_2 x_2(t)) \neq \alpha_1 Dx_1 + \alpha_2 Dx_2 \quad \text{[D nonlinear]} \]

\[ \omega_0 = \sqrt{\frac{g}{L}} \text{ the "natural frequency" and } \gamma = \frac{F_0}{mL \omega_0^2} \text{ the "drive strength".} \]

\( \gamma \) is dimensionless (by design).

Since \( F_0 / mL \) has dimension \( \frac{\text{time}}{\text{time}^2} \).

The name makes sense because \( \gamma = \frac{F_0}{mL \omega_0^2} = \frac{F_0}{mg} \)

for \( \gamma \leq 1 \), \( F_0 < mg \) and has a small effect, while for \( \gamma \geq 1 \), \( F_0 > mg \) and has a large effect.
Finally then,
\[ \ddot{\phi} + 2\beta \dot{\phi} + \omega_0^2 \sin \phi = \gamma \omega_0^2 \cos \omega t \]

To appreciate how wild chaos is, we first need to remind ourselves of what we would naively expect.

For small oscillations,
\[ \ddot{\phi} + 2\beta \dot{\phi} + \omega_0^2 \phi = \gamma \omega_0^2 \cos \omega t \]
we get a damped driven oscillator!

If we now increase the drive strength the amplitude will increase and more terms will matter: \[ \sin \phi \approx \phi - \frac{1}{6} \phi^3 \]
\[ \ddot{\phi} + 2\beta \dot{\phi} + \omega_0^2 (\phi - \frac{1}{6} \phi^3) = \gamma \omega_0^2 \cos \omega t \]
This is a small perturbation, so guess \( \phi(t) \approx A \cos(\omega t - \delta) \)

When we put this guess into the E.O.M. we get a term \( \phi^3 = A^3 \cos^3(\omega t - \delta) \). A useful trig identity is

We expect that for weak driving \( \frac{\beta}{\omega_0^2} \) and small oscillations the initial behavior of the pendulum depends on the initial conditions, but this transient motion dies out rapidly, and the proton approaches a unique attractor: \( \phi(t) = A \cos(\omega t - \delta) \)

see plot (Fig 1).

\[ \cos^3 x = \frac{1}{4} (\cos 3x + 3 \cos x) \]

Then a simplification of the \( \phi^3 \) term will include \( \cos(3[\omega t - \delta]) \) but the drive has a prescribed frequency \( \omega \) and so this tripled frequency must be cancelled by one of \( \phi, \phi, \dot{\phi} \) (in fact all three), and a better guess is

\[ \phi(t) = A \cos(\omega t - \delta) + B \cos 3(\omega t - \delta) \]
As the drive increases this pattern continues and we get higher harmonics (multiples of \( (\omega t - S) \)). This slowly modifies the attractor solution (see Fig. 2) as we approach \( \gamma = 1 \). Near \( \gamma = 1 \), we get dramatically different results!

III Midterm 2 Histogram & Results.
Figure: 1) Taken from Taylor p464
Figure: 2) Taken from Taylor p466