Today's Outline

I Last Lecture
& finish double pendulum

II Lagrangian Approach:
the general case

Lecture 29
November 7th, 2011
We studied the double pendulum and found the E.O.M.

\[(m_1 + m_2) l_1^2 \ddot{\phi}_1 + m_2 l_1 l_2 \ddot{\phi}_2 = -(m_1 + m_2) g l_1 \phi_1\]

\[m_2 l_1 l_2 \ddot{\phi}_1 + m_2 l_2^2 \ddot{\phi}_2 = -m_2 g l_2 \phi_2\]

These E.O.M are equivalent to

\[\ddot{\vec{M}} \phi = -K \phi\]

with \(\phi = (\phi_1, \phi_2)\) and

\[\vec{M} = \begin{pmatrix} (m_1 + m_2) l_1^2 & m_2 l_1 l_2 \\ m_2 l_1 l_2 & m_2 l_2^2 \end{pmatrix}, \quad K = \begin{pmatrix} (m_1 + m_2) g l_1 & 0 \\ 0 & m_2 g l_2 \end{pmatrix}\]

In this form the problem is exactly like our toy model:

Guess: \(\hat{\phi}(t) = R e^{i \omega t}, \quad \hat{\phi}'(t) = \hat{a} e^{i \omega t} = (a_1, a_2) e^{i \omega t}\)

Solve: \(\det(\vec{K} - \omega^2 \vec{M}) = 0\) for normal modes

\(\begin{pmatrix} \lambda_1 - \omega^2 \vec{M} \end{pmatrix} \phi = 0\) for normal modes

For example when \(m_1 = m_2 = m\) and \(l_1 = l_2 = l\) and we let \(\omega_0 = \sqrt{g/l}\)
then

\[\omega_1 = \sqrt{2 - \sqrt{2}} \omega_0 \approx 0.77 \omega_0\]

\[\omega_2 = \sqrt{2 + \sqrt{2}} \omega_0 \approx 1.85 \omega_0\]

\[\hat{a}_1 = \left(\frac{1}{\sqrt{2}}\right) A_1 e^{i \delta_1}, \quad \hat{a}_2 = \left(-\frac{1}{\sqrt{2}}\right) A_2 e^{i \delta_2}\]
II Lagrangian Approach: the general case

A system of \( n \) degrees of freedom oscillating about a point of stable equilibrium. We will describe the system (assumed holonomic) by \( n \) generalized coordinates \( \varphi_1, \ldots, \varphi_n \) or \( \tilde{\varphi} = (\varphi_1, \ldots, \varphi_n) \) (e.g. \( \tilde{\varphi} = (x, y) \) for toy model or \( \tilde{\varphi} = (\phi_1, \phi_2) \) for double pendulum).

Then

\[
T = \frac{1}{2} \sum_{\alpha} m_{\alpha} \dot{r}_{\alpha} \cdot \dot{r}_{\alpha}
\]

\[
= \frac{1}{2} \sum_{\alpha} m_{\alpha} \left( \sum_{i,j} \ddot{\varphi}_i \ddot{\varphi}_j \right) \cdot \left( \sum_{k} \ddot{\varphi}_k \ddot{\varphi}_k \right)
\]

\[
= \frac{1}{2} \sum_{i,j,k} A_{ijk} \ddot{\varphi}_i \ddot{\varphi}_j \ddot{\varphi}_k
\]

with

\[
A_{ijk} = \sum_{\alpha} m_{\alpha} \left( \frac{\partial \tilde{r}_{\alpha}}{\partial \varphi_i} \right) \cdot \left( \frac{\partial \tilde{r}_{\alpha}}{\partial \varphi_j} \right) \cdot \left( \frac{\partial \tilde{r}_{\alpha}}{\partial \varphi_k} \right)
\]

\[
A_{ijk} = A_{ijk}(\varphi_1, \ldots, \varphi_n) = A_{ijk}(\tilde{\varphi})
\]

As always the K.E. is

\[
T = \frac{1}{2} \sum_{\alpha} m_{\alpha} \dot{r}_{\alpha} \cdot \dot{r}_{\alpha} \quad \alpha = (1, \ldots, n)
\]

We write this in terms of the generalized coordinates by using the transformation

\[
\tilde{r}_{\alpha} = \tilde{r}_{\alpha}(\varphi_1, \ldots, \varphi_n)
\]

By the chain rule

\[
\dot{\tilde{r}}_{\alpha} = \sum_{i=1}^{n} \frac{\partial \tilde{r}_{\alpha}}{\partial \varphi_i} \dot{\varphi}_i
\]

We assume the forces are conservative so that

\[
U = U(\varphi_1, \ldots, \varphi_n) = U(\tilde{\varphi})
\]

Next, we assumed \( \tilde{\varphi}_0 \) is a stable equilibrium and by translating the origin take \( \tilde{\varphi}_0 = \tilde{0} \). For small oscillations we can Taylor expand

\[
U(\tilde{\varphi}) = U(0) + \sum_{j} \left( \frac{\partial U}{\partial \varphi_j} \right)_{\varphi=0} \tilde{\varphi}_j + \frac{1}{2} \sum_{i,j} \left( \frac{\partial^2 U}{\partial \varphi_i \partial \varphi_j} \right)_{\varphi=0} \tilde{\varphi}_i \tilde{\varphi}_j
\]

\( \leq \) constant which can be dropped.

\[+\cdots\]
If we let \( K_{jk} \equiv \frac{\partial^2 U}{\partial \phi_j \partial \phi_k} \mid_{\phi = \bar{\phi}} \) (note that \( K_{jk} = K_{kj} \)) and neglect higher order terms, we have,

\[
U = U(\bar{\phi}) = \frac{1}{2} \sum_{j,k} K_{jk} \dot{\phi}_j \dot{\phi}_k
\]

Actually, since we're dropping higher order terms already of 2nd order,

\[
T = \frac{1}{2} \sum_{j,k} \sum_{\ell} A_{jk}(\bar{\phi}) \dot{\phi}_j \dot{\phi}_k \approx \frac{1}{2} \sum_{j,k} M_{jk} \dot{\phi}_j \dot{\phi}_k
\]

gives solvable linear E.O.M. out of 'em. Let's do it:

\[
\mathbf{x} = T - U
\]

\[
\frac{\partial \mathbf{x}}{\partial \dot{\phi}_i} = \frac{1}{2} \left( \frac{1}{2} \sum_{j,k} M_{jk} \dot{\phi}_j \dot{\phi}_k \right) - \partial U
\]

\[
\frac{\partial \mathbf{x}}{\partial \phi_i} = \frac{1}{2} \left( \sum_{j,k} M_{jk} \dot{\phi}_j \dot{\phi}_k + \sum_{j,k} M_{jk} \dot{\phi}_j \dot{\phi}_k \right)
\]

\[
\frac{\partial \mathbf{x}}{\partial \phi_i} = \frac{1}{2} \left( \sum_{j,k} M_{jk} \dot{\phi}_j \dot{\phi}_k + \sum_{j,k} M_{jk} \dot{\phi}_j \dot{\phi}_k \right) = \sum_{j} M_{ij} \dot{\phi}_j
\]

\( \Rightarrow \dot{\mathbf{M}} \ddot{\phi} = -\mathbf{K} \ddot{\phi} \)

Where

\[
M_{jk} = A_{jk}(\bar{\phi})
\]

is a matrix of constants.

Note that this simplifies \( T \) so that \( T = T(\bar{\phi}, \dot{\phi}) \) reduces to \( \dot{T} = T(\bar{\phi}) \).

What have we achieved? Once again \( T \) and \( U \) are homogeneous quadratic functions of the \( \dot{\phi}s \) and \( \phi \)s respectively! Again we'll

\[
\frac{d}{dt} \left( \frac{\partial \mathbf{x}}{\partial \dot{\phi}_i} \right) = \sum_{j} M_{ij} \ddot{\phi}_j
\]

\[
\frac{\partial \mathbf{x}}{\partial \phi_i} = \frac{\partial T}{\partial \phi_i} - \frac{\partial U}{\partial \phi_i}
\]

\[
= -\frac{1}{2} \frac{\partial}{\partial \phi_i} \left( \sum_{j,k} K_{jk} \dot{\phi}_j \dot{\phi}_k \right)
\]

\[
= -\sum_{j} K_{ij} \ddot{\phi}_j
\]

\( \Rightarrow \dot{\mathbf{M}} \ddot{\phi} = -\mathbf{K} \ddot{\phi} \)
Again we guess,
\[ \hat{\mathbf{x}}(t) = \text{Re} \, \hat{\mathbf{Z}}(t) \] with \[ \hat{\mathbf{Z}}(t) = \hat{\mathbf{a}} e^{i \omega t} \]
and solve:
\[
\det(\hat{\mathbf{K}} - \omega^2 \hat{\mathbf{M}}) = 0 \quad \text{normal freqs.}
\]
\[
(\hat{\mathbf{K}} - \omega^2 \hat{\mathbf{M}}) \hat{\mathbf{a}} = 0 \quad \text{normal modes}
\]
The det equation is an Nth degree polynomial in \( \omega^2 \) which we solve for the N normal frequencies. Then we determine the N \( \hat{\mathbf{a}} \) vectors. And

Finally, the general solution \( \Phi(t) \) is a linear combination of the \( N \) normal modes.

Notice that in fact you needn't write down the Lagrangian at all for these problems any more; you can just find \( \hat{\mathbf{M}} \) and \( \hat{\mathbf{K}} \) from the general formulæ we wrote down.

Continuum Mechanics

We've completed the core of the course:

- Oscillations
- Calculus of Variations
- Lagrangian Mechanics
- Central Forces
- Noninertial Frames
- Rigid Bodies

Normal Modes
- Special Topics:
  - (to come)
- Nonlinear Mechanics
- Chaos
- Hamiltonian Mechanics