Today's Outline

I. Last Lecture

II. Free-fall with $F_{cor}$

III. Foucault's pendulum

Lecture 20

October 14th, 2011

I. Last Lecture

We carefully worked out:

- the direction of the centripetal force
- the impact of $F_{cs}$ on $\ddot{q}$.
- The direction of the Coriolis force (in both the Northern and the Southern hemispheres).

II. Free-fall again

$$\ddot{r} = mg + F_{cs} + F_{cor}$$

$$= m\ddot{g} + 2m\dddot{r} \times \dddot{r}$$

$$\Rightarrow \dddot{r} = \dddot{g} + 2\dddot{r} \times \dddot{r}$$

Notice that this equation only depends on $\dddot{r}$ and $\dddot{r} \Rightarrow$ we can arbitrarily shift our origin (see figure):

Cross-section of Earth:

In these coords,

$$\dddot{r} = (\dddot{x}, \dddot{y}, \dddot{z})$$

and

$$\dddot{r} = (0, a\sin\theta, a\cos\theta)$$

so that

$$\dddot{r} \times \dddot{r} = \begin{vmatrix} \dot{x} & \dot{y} & \dot{z} \\ x & y & z \\ 0 & a\sin\theta & a\cos\theta \end{vmatrix} = (\dddot{y} \Omega \cos\theta - \dddot{z} \Omega \sin\theta, -\dddot{z} \Omega \cos\theta, \dddot{x} \Omega \sin\theta)$$
The E.O.M. is then

\[ \ddot{x} = 2\Omega (y \cos \theta - \dot{z} \sin \theta) \]
\[ \ddot{y} = -2\Omega \dot{x} \cos \theta \]
\[ \ddot{z} = -g + 2\Omega \dot{z} \sin \theta \]

which is a set of coupled differential equations. Not easy to solve, so we will make a series of approximations.

First \( \Omega \ll 1 \), so let's take \( \Omega \approx 0 \)

\[ \Rightarrow \dot{x} = 0 \quad \dot{y} = 0 \quad \text{and} \quad \ddot{z} = -g \]

First order is then

\[ x(t) = \frac{1}{3} \Omega g t^3 \sin \theta \quad y = 0 \quad z = h - \frac{1}{2} gt^2 \]

This reasonably has only \( \Omega^0 = 1 \) and \( \Omega^1 = \Omega \) in it. We could continue in this manner to get as many powers of \( \Omega \) as we wanted. How big is this effect?

Drop a pebble down a 100 meter mine shaft at the equator and

with solution,

\[ x = 0 \quad y = 0 \quad z = h - \frac{1}{2} gt^2 \]

assuming \( z(0) = h, \dot{z}(0) = 0 \). This is called the zeroth order approximation because it only has \( \Omega^0 = 1 \) in it. To get the first order we put the zeroth order back into the E.O.M. to find

\[ \dot{x} = 2\Omega g t \sin \theta \quad \dot{y} = 0 \quad \ddot{z} = -g \]

\[ \Rightarrow \dot{x}(t) = 2\Omega g t^2 \sin \theta + y^0 \Rightarrow x(t) = \frac{1}{3} \Omega g t^3 \sin \theta + y^0 \]

we have \( z = h - \frac{1}{2} gt^2 \Rightarrow t = \sqrt{\frac{2h}{g}} \). while

\[ \dot{x} = \frac{1}{3} \Omega g \left( \frac{2h}{g} \right)^{3/2} \approx 2.2 \text{ cm}, \]

generally a small effect.

This example illustrates:

- how to calculate cross-products (a reminder)
- one approach to solving coupled differential equations.
III Foucault's pendulum

Foucault's pendulum is a spherical pendulum (like your HW problem) with a massive bob and a long wire. The pendulum is suspended from a pivot $P$, fixed to the earth.

\[ T \]

\[ \hat{x} \]

\[ \hat{y} \]

\[ \hat{z} \]

\[ x \text{ (east)} \]

\[ y \text{ (north)} \]

\[ z \text{ (up)} \]

\[ m \]

\[ L \]

The E.O.M. for the pendulum is

\[ \ddot{x} + \frac{L}{T} \dot{y} + \frac{L}{T} \dot{z} + \frac{2}{T} \dot{\hat{z}} \times \hat{x} = 0 \]

\[ \ddot{y} - \frac{L}{T} \dot{x} - \frac{L}{T} \dot{z} + \frac{2}{T} \dot{\hat{z}} \times \hat{y} = 0 \]

Consider case where $\beta$ is small so that

\[ T_2 = T \cos \beta \approx T \]

For small $\beta$ we also have $\hat{z}$ and $\dot{\hat{z}}$ small and the $z$-component of the E.O.M. becomes,

\[ \theta = T_2 - mg \Rightarrow T_2 \approx T \times Mg \]

Similarly

\[ T_2 = -T \frac{y}{L} = -Mg \frac{y}{L} \]

Putting it together our E.O.M are

\[ \ddot{x} = -g \frac{x}{L} + 2 \dot{y} \Omega \cos \theta \]

\[ \ddot{y} = -g \frac{y}{L} - 2 \dot{x} \Omega \cos \theta \]

Noting that $g \frac{y}{L} = \omega_0$ and $\Omega \cos \theta = \Omega \hat{z}$

we have

\[ \ddot{x} = 2 \Omega \dot{y} + \omega_0^2 x = 0 \]

\[ \ddot{y} + 2 \Omega \dot{x} + \omega_0^2 y = 0 \]

Another set of coupled equations!
These are almost harmonic oscillator equations, we'll use another new technique to solve them. Let

\[ \eta = x + iy \]

and multiply the \( \dot{x} \) equation by \( i = \sqrt{-1} \) and add it to the \( \dot{y} \) equation, to find

\[ \ddot{x} + 2i \omega_z \dot{x} + \omega^2 x = 0 \]

This is a 2nd order, linear, homogeneous diff. eq. We can go back to our original equation:

Recall \( \omega_z < \omega_0 \) and so

\[ \lambda \approx -i \left( \omega_z \pm i \omega_0 \right) \]

Our general solution is then

\[ \eta(t) = C_1 e^{-i(\omega_z - i\omega_0)t} + C_2 e^{-i(\omega_z + i\omega_0)t} \]

\[ = e^{-i\omega_z t} \left[ C_1 e^{i\omega_0 t} + C_2 e^{-i\omega_0 t} \right] \]

To set \( C_1 \) and \( C_2 \) we need initial conditions, let's choose: \( x(0) = A \) \( y(0) = 0 \)

Standard guess:

\[ \eta(t) = e^{\lambda t} \]

(Note: notation differs from book but agrees with lecture 1.)

\[ \Rightarrow \lambda^2 + 2i \omega_z \lambda + \omega_0^2 = 0 \]

\[ \Rightarrow \lambda = -z \omega_z \pm \sqrt{-4 \omega_z^2 - 4\omega_0^2} \]

\[ = -i \left( \omega_z \pm i \omega_0 \right) \]

(5) book's \( \lambda = \) our \( i \lambda \).

\[ u_{x_0} = u_{y_0} = 0 \] then \( \eta(0) = A \) and \( \dot{\eta}(0) = 0 \) but

\[ \eta(0) = C_1 + C_2 \]

\[ \dot{\eta}(0) = -i(\omega_z - i\omega_0)C_1 - i(\omega_z + i\omega_0)C_2 \]

\[ = i\omega_0 C_1 - i\omega_0 C_2 \]

Then, \( C_1 + C_2 = A \) \( C_1 - C_2 = 0 \)

\[ \Rightarrow C_1 = C_2 = A/2. \]

and

\[ \eta(t) = x + iy = A e^{-i\omega_z t} \cos(\omega_0 t) \]

usual pendulum motion