Reading Assignment: Read pp. 60–70 of the book, and study lecture notes for Mar. 12. You can skip Sec. 4.13 of the book, however, since it is pretty hard to follow. In particular, look at Eq. (4.30) in the book. On the left hand side we have the product of two group elements, while on the right hand side we have a linear combination of group elements. You can’t add group elements in general, so the right hand side makes no sense. What the authors of the book really mean by this equation is explained in the lecture notes (the equation refers to the induced transformation acting on what was called the Δ-basis in the notes). Also there is a misprint just above Eq. (4.35), where it says, “i.e., \( v_a = v_a \),” when it means \( v_c = v_a \). We did not cover the material in Sec. 4.14 in lecture last Wednesday, but we will do it next Wednesday.

Our main accomplishment this week was to derive a number of orthogonality relations that follow from the G. O. T. The following is a summary of these.

First, the G. O. T. itself:

\[
\frac{1}{\#G} \sum_g M_{ij}^{(r)}(g)^* M_{k\ell}^{(s)}(g) = \frac{1}{d_r} \delta_{rs} \delta_{ik} \delta_{j\ell},
\]

where \( r, s \) label irreps, where the matrices \( M \) are assumed to be unitary, and where \( d_r \) is the dimension of irrep \( r \). This is a statement of the orthogonality of the rows of a table of matrix elements of the irreps, such as the table on p. 12 of the lecture notes for 3/5/03 for the group \( C_3v \). It takes some work with the regular representation to show this table has an equal number of rows and columns, that is,

\[
\sum_r d_r^2 = \#G,
\]

but once this is done, the table is seen to be square and forms a unitary matrix (if the rows are normalized). Thus, the columns also obey an orthonormality relation, which is

\[
\frac{1}{\#G} \sum_{rij} d_r M_{ij}^{(r)}(g)^* M_{k\ell}^{(r)}(h) = \delta_{gh},
\]

This can be thought of as the inverse of the G. O. T.
A consequence of these orthonormality and completeness relations is that any function \( \psi(g) \) on the group can be expanded as a linear combination of the matrix elements of the irreps, that is, an expansion of the form

\[
\psi(g) = \sum_{r \in \mathcal{R}} c_{rij} M^{(r)}_{ij}(g)
\]

for all functions \( \psi(g) \), and the expansion coefficients are given by

\[
c_{rij} = \frac{d_r}{\# G} \sum_g M^{(r)}_{ij}(g)^* \psi(g).
\]

Next, we define the character of any representation \( g \mapsto M(g) \) (reducible or not) by

\[
\chi(g) = \text{tr} M(g),
\]

and we write

\[
\chi^{(r)}(g) = \text{tr} M^{(r)}(g)
\]

in the case it is an irrep. We note that \( \chi \) is a class function, that is, it is constant within any conjugacy class. This leads to the character table of a group, exemplified on p. 4 of the lecture notes for 3/12/03 for the group \( C_{3v} \), whose rows are labelled by the irreps of the group and whose columns are labelled by the conjugacy classes. It follows by taking traces in the G. O. T. that the character table obeys a kind of orthogonality of its rows, namely,

\[
\frac{1}{\# G} \sum_C (\# C) \chi^{(r)}(C)^* \chi^{(s)}(C) = \delta_{rs},
\]

where \( C \) is an index of the conjugacy classes of \( G \). It turns out that the rows of the character table are also complete, that is, any class function can be expanded as a linear combination of the characters of the irreps. We proved this in class by working with the completeness of the matrix elements of the irreps. This means that the number of rows of the character table is equal to the number of its columns, or,

\[
\# \text{ irreps} = \# \text{ classes}.
\]

It also means that the character table satisfies a kind of orthogonality by columns, which is

\[
\frac{\# C}{\# G} \sum_r \chi^{(r)}(C)^* \chi^{(r)}(C') = \delta_{CC'}.
\]

We didn’t prove this last identity in class (we will do it next Wednesday).
One of the most important applications of character tables is to finding the multiplicity of various irreps contained in a given, possibly reducible, representation. Let \( g \mapsto M(g) \) be a given representation, with characters \( \chi(g) \). As we showed in class,

\[
\chi(g) = \sum_r \mu_r \chi^{(r)}(g),
\]

where \( \mu_r \) is the multiplicity of irrep \( r \) in \( M(g) \). This comes from taking the trace of \( M(g) \) in a basis that block diagonalizes it (and the trace doesn’t depend on the basis anyway). The multiplicities can be determined by using the orthogonality of the rows of the character table, which leads to

\[
\mu_r = \frac{1}{\#G} \sum_C (\#C) \chi^{(r)}(C)^* \chi(C).
\]

We also derived the formula,

\[
\frac{1}{\#G} \sum_C (\#C) |\chi(C)|^2 = \sum_r \mu_r^2,
\]

which leads to a simple criterion for the irreducibility of \( g \mapsto M(g) \): it is irreducible iff the right hand side equals 1. (And it is simple because the computation does not require you to know anything about the irreps).

Now for the homework problems.

1. On pages 2–3 of the lecture notes for 2/26/03 is listed a 6-dimensional representation of \( C_{3v} \). Find the multiplicities \( \mu_r \) of the different irreps of \( C_{3v} \) in this 6-D representation. This representation is not unitary. Do you need to worry about Maschke’s theorem in the calculation of \( \mu_r \)? Also check Eq. (13) for this 6D rep.

2. Find the conjugacy classes for the group \( D_4 \) (see problem 1 of HW 2). Be careful, some students did not compute conjugacy classes correctly in problem 2 of HW 3. Next use the hints in Sec. 4.15 of the book to find the character table for \( D_4 \). Use also this hint: In any one-dimensional irrep, the character \( \chi(g) \) of a group element \( g \) is one of the \( n \)-th roots of unity, where \( n \) is the order of the group element \( g \). Why is this true?