
Forgot to mention: In problem 1 of last week, I asked you to prove that the eigenspaces of a normal operator are orthogonal. I forgot to mention that this is part of the proof that normal matrices can be diagonalized by a unitary matrix. The rest of the proof is to show that normal matrices are complete (they possess a complete set of eigenvectors). You knew already that Hermitian matrices can be diagonalized by a unitary transformation; these theorems show that the same is true for any normal matrix. For example, any unitary matrix can be diagonalized by a (another) unitary matrix. In this case, the eigenvalues on the diagonal of the diagonal matrix are phase factors.

1. Let $g \mapsto M(g)$ be a representation of a finite group by means of $n \times n$ matrices $M(g)$. According to Maschke’s theorem, there is no loss of generality in assuming that the matrices $M(g)$ are unitary. The representation is said to be irreducible if it possesses no nontrivial, invariant subspace. The trivial subspaces have dimension 0 or $n$; all the rest are nontrivial. Sometimes it is possible to prove irreducibility working directly from this definition.

   (a) Show that all $1 \times 1$ representations of a group are irreducible. In particular, the trivial representation is irreducible.

   (b) In class we discussed the $3 \times 3$, orthogonal representation of the group $C_{3v}$ (see p. 1 of the lecture notes for 2/26/03 for the matrices). It was pointed out that this representation is reducible, as you can see by looking at the matrices, which are block diagonal. The invariant subspaces are the $x$-$y$ plane and the $z$-axis. The upper $2 \times 2$ block of these matrices gives us a $2 \times 2$ representation of $C_{3v}$. Show that this representation is irreducible. Hint: These representations happen to be real, but you must work with complex numbers and complex vector spaces.

2. About the irreducible representations of Abelian groups. Here we suppose the group is of finite order, and we are thinking of matrix representations, $g \mapsto M(g)$.

   (a) Show that all irreducible representations of an Abelian group are 1-dimensional. Hint: First show that every eigenspace of every matrix $M(g)$ is an invariant subspace of the whole
set of matrices. This fact implies that degeneracies in quantum mechanics are associated with non-Abelian symmetry groups.

(b) A one-dimensional representation of a group is obviously just an association between group elements $a$ and numbers $f(a)$, such that $f(a)f(b) = f(ab)$ (it is a function $f : G \to \mathbb{C}$). Show that two one-dimensional representations are equivalent if and only if the corresponding functions are identical.

(c) The group $C_n$ is the cyclic group of order $n$, $(e, a, a^2, \ldots, a^{n-1})$, with $a^n = e$. It is Abelian. Find all the inequivalent, irreducible representations of this group. Hint: There are $n$ of them.