Reading Assignment: Read pp. 17–30 of the book, and study lecture notes for Feb. 5.

1. In physical problems groups usually arise as transformation groups, that is, groups of transformations on some space. For example, there are rotations acting on ordinary 3-dimensional space, Lorentz transformations acting on space-time, and various transformations on spaces of wave functions.

Let $S$ be some space. By a transformation we mean an invertible mapping of this space onto itself, $T : S \rightarrow S$ (something that makes points get up and move to other points, in such a manner that no two points end up at the same point). Now let $G$ be a group. An action of $G$ on $S$ is an association between elements $a$ of $G$ and transformations $T_a$ such that

$$T_a T_b = T_{ab}, \quad \text{for all } a, b \in G. \quad (1)$$

Thus, the transformations reproduce the group multiplication law.

Now let $x$ be a point in $S$, and suppose we have the action of some group $G$ on $S$. Then the set of points that can be reached from $x$ by applying transformations $T_a$ to $x$ is called the orbit of $x$ (under the specified group action). (This is unfortunate terminology, since those who have not studied group theory will think you are talking about orbits of particles in classical mechanics. Actually, the orbit of a particle in classical mechanics can be interpreted as an orbit in this group theoretical sense.) We will write $O_x$ for the orbit of $x$; it is a subset of $S$ given by

$$O_x = \{T_a x | a \in G\}. \quad (2)$$

(a) Show that the group action breaks the space $S$ into disjoint subsets (the orbits), that is, show that if two orbits have one point in common then they have all points in common, otherwise they are disjoint.

Some group elements may not do anything to $x$. For example, consider rotations about the $z$-axis in 3-dimensional space. These rotations do not move the points on the $z$-axis. Let $x \in S$ be a point. Define $I_x$ as the set of elements of $G$ that do nothing to $x$, that is,

$$I_x = \{a \in G | T_a x = x\}. \quad (3)$$
(b) Show that \( I_x \) is a subgroup of \( G \). It is called the *isotropy subgroup* or *stabilizer* of \( x \). Different points \( x \) have different stabilizers, in general. Show that the points of the orbit \( O_x \) can be placed in one-to-one correspondence with the left cosets \( aI_x \) of the stabilizer of \( x \). Conclude therefore that if \( G \) is of finite order, then the number of points in an orbit is a divisor of \( \#(G) \).

(c) A special case of an action is that of \( G \) on itself (\( G = S \) in this example), defined by \( T_a = a g \). This is the action of left multiplication. What is the stabilizer \( I_g \) in this case? What are the orbits? A slight generalization of this is the following. Let \( H \) be a subgroup of \( G \). We then define an action of \( H \) on \( G \) by \( T_a g = a g \), where now \( a \in H \). What is the stabilizer \( I_g \) in this case? What are the orbits? Finally, here is another action of \( G \) on itself, \( T_a g = a g a^{-1} \) (the action is conjugation). Show that this really is an action (that Eq. (1) is satisfied). What are the orbits in this case? Explain why the number of elements in a conjugacy class is a divisor of the number of elements in the group. For an example of this, we showed in class that the conjugacy classes of the group \( D_3 \) were \( \{E\} \), \( \{R_1, R_2\} \), and \( \{R_3, R_4, R_5\} \) (see p. 6 of the lecture notes for 2/5/03).

(d) If \( H \) is a subgroup of \( G \) and \( a \in G \), we define the set

\[
aH a^{-1} = \{aha^{-1}|h \in H\}. \tag{4}
\]

This set is obtained by conjugating every element in \( h \) by \( a \). Show that this set is a subgroup of \( G \). It is said to be a subgroup *conjugate* to \( H \). Show that conjugate subgroups are isomorphic. In physical problems, conjugate subgroups often represent the “same physics,” just seen by “different observers.” Consider the subgroup \( \{E, R_3\} \) in the group \( D_3 \). Find all the distinct conjugate subgroups. Finally, let \( S \) be a space upon which \( G \) has an action. Let \( x \) and \( y \) be two points on the same orbit \( O_x \) of the group action. Do these points have the same stabilizer?

![Fig. 1. A tetrahedron.](image-url)
2. Consider the tetrahedron (see the figure). The group of proper covering operations is called the tetrahedral group (naturally). It has four 3-fold axes that pass through the center of a face on one side and a vertex on the other. It also has three 2-fold axes, passing through the centers of two edges each. The total number of group elements is 12.

(a) Find the conjugacy classes of the tetrahedral group. Do not try to write out a group multiplication table, instead use the formula

\[ R_0R(\hat{n}, \theta)R_0^{-1} = R(R_0\hat{n}, \theta), \]

discussed in class. You will have to understand the definition of conjugacy classes and the meaning of this formula. Describe your conjugacy classes this way: “Class one contains all rotations by angles blah-blah and blah-blah about the such-and-such symmetry axes.” Make sure you specify the number of elements in each class.

(b) Find and describe the conjugacy classes of the icosahedral group. You may want to refer to your dodecahedron.