(Continuing with SO(2) symmetry in a 2D central force problem in GM).

What is the symmetry adapted basis for this problem (rather should say, a symm. adapt. basis, since it's not unique)? Previously we wrote $|r\alpha i\rangle$ for the basis vectors of the S.A.B. Now the Dirac notation is appropriate, since we are talking about wavefunctions (quantum states). However, now the index $r$ is replaced by $\mu$ and $\alpha$ is replaced by $m$ (the index of the radial basis for $u_\mu(r)$). The index $i$ is not needed since the irreps are 1D. So the S.A.B. is

$$\{|\mu m\rangle\} = \{\text{e}^{-i\phi}u_\mu(r)\}.$$ 

One final note on this, usually it's appropriate to choose a different basis of radial functions for each value of $\mu$, that is $u_\mu(r)$ becomes $\text{Un}m(r)$.

Now let's turn to SO(3). We know a lot about this group already.

We know that it is non-Abelian and that every $R \in SO(3)$ can be written in axis-angle form, $R(\mathbf{n}, \theta)$, for some $\mathbf{n}, \theta$. Let us begin by figuring out what the group manifold is for SO(3). It's clearly a 3D manifold, since there are 3 parameters involved in the axis $\mathbf{n}$ and angle $\theta$. (Two for $\mathbf{n}$, one for $\theta$.) The only 3D manifold that's easy to visualize is $\mathbb{R}^3$, because that's the space we live in (or the space we usually think we live in). But with a little imagination,
It's not hard to picture the group manifold $SO(3)$. We just have to work with the parameters $(\hat{n}, \theta)$ and figure out to what extent this parametrization is one-to-one.

First recall the identity,

$$R(\hat{n}, 2\pi - \theta) = R(-\hat{n}, \theta),$$

so we must restrict the angle to the range $0 \leq \theta \leq \pi$ if we want $\hat{n}$ to run all over the unit sphere. Take the case $0 < \theta < \pi$, and fix a value of $\theta$ in this range. Then all rotations with this $\theta$ are obtained by letting $\hat{n}$ run over the unit sphere, so the space of all rotations with a fixed $\theta$, $0 < \theta < \pi$, is the 2D surface of a sphere, usually denoted $S^2$. Similarly, the space of rotations with $\theta_1 \leq \theta \leq \theta_2$, where $0 < \theta_1 < \theta_2 < \pi$, is a spherical shell where the "radius" is $\theta$ and the direction from a center to a point is $\hat{n}$:

![Diagram showing spherical shell with radius $\theta$ and direction $\hat{n}$]

This is equivalent to introducing coordinates $\vec{\theta} = \hat{n} \theta$, a 3-vector of angles. Now take $\theta_1 \to 0$. At $\theta = 0$, $R(\hat{n}, \theta)$ consists of just a single rotation, since $R(\hat{n}, 0) = I$ for all $\hat{n}$. 
This is what we get automatically if we let the inner radius go to zero in the sphere picture (a single point at the origin represents the identity). Finally, let $\theta_2 = \pi$. The angle $\theta = \pi$ is tricky. The set of all rotations with angle $\pi$ is not a sphere (unlike other angles $0 < \theta < \pi$), because by the identity above we have

$$R(n, \pi) = R(-n, \pi)$$

So at radius $\theta = \pi$, antipodal points represent the same rotation.

So the group manifold $SO(3)$ is (topologically speaking) the solid interior of a sphere in 3D space, plus the surface, with the rule that antipodal points on the surface are considered the same point. (This surface is also called $\mathbb{R}P^3$ in mathematical notation, the 3D real projective space.)
Now let's look at rotations $R(\hat{n}, \theta)$ for fixed $\hat{n}$. These are rotations about an axis $\hat{n}$ and it's obvious geometrically that such rotations commute,

$$R(\hat{n}, \theta_1) R(\hat{n}, \theta_2) = R(\hat{n}, \theta_1 + \theta_2) = R(\hat{n}, \theta_2) R(\hat{n}, \theta_1).$$

In fact, the set

$$\{ R(\hat{n}, \theta) \mid 0 \leq \theta < 2\pi \}$$

forms an Abelian subgroup of $SO(3)$ which is isomorphic to $SO(2)$. In fact, it's just $SO(2)$ rotations in a plane perpendicular to $\hat{n}$.

These rotations constitute a 1-parameter subgroup of $SO(3)$. Geometrically, inside the group manifold this subgroup is a straight line passing through the origin.

Since antipodal points on the surface are identified, this line segment is really a circle, the group manifold $SO(2)$. (topologically speaking)
Previously we studied the formula,
\[ R_0 R(\hat{n}, \theta) R_0^{-1} = R(R_0 \hat{n}, \theta), \]
valid for \( R_0 \in \text{SO}(3) \) and \( R(\hat{n}, \theta) \in \text{SO}(3) \). We'll prove this formula shortly.

For now notice that it says that the various 1-parameter subgroups \( \{ R(\hat{n}, \theta) \mid 0 \leq \theta < 2\pi \} \) (which you get by fixing \( \hat{n} \)) are conjugate to one another. The \( R_0 \) rotates the old axis into the new one. In the sphere (or \( \mathbb{RP}^3 \)) picture of the group manifold, the line through the origin specifying the 1-param subgroup is rotated by \( R_0 \) into a new direction.

Let's go back to ordinary 3D space, upon which \( \text{SO}(3) \) acts. Draw a picture of the action of \( R(\hat{n}, \theta) \) on some vector \( \vec{u} \). Obviously \( R(\hat{n}, \theta) \) does not affect the component of \( \vec{u} \) parallel to \( \hat{n} \), while the component \( \perp \hat{n} \) gets rotated like a vector in a plane under \( \text{SO}(2) \).

![Diagram of rotation](attachment:rotation_diagram.png)

\[ \vec{u}_1 = \hat{n} (\hat{n} \cdot \vec{u}) \]

\( \vec{u} \) sweeps out a cone as \( \theta \) increases in \( R(\hat{n}, \theta) \).
\[
\hat{u} = \hat{u}_{\parallel} + \frac{(\hat{n} \times \hat{u}) \times \hat{n}}{\text{comp. } \bot, \perp \text{ to } \hat{n}} \quad \text{expand}, \quad \hat{u} - \hat{n}(\hat{n} \cdot \hat{u}) = \hat{u}_\perp
\]

So,

\[
R(\hat{n}, \theta)\hat{u} = R(\hat{n}, \theta)\hat{u}_{\parallel} + R(\hat{n}, \theta)\hat{u}_\perp
\]

\[
\Rightarrow = \hat{u}_{\parallel}
\]

\[
\Rightarrow = \cos \theta [(\hat{n} \times \hat{u}) \times \hat{n}] + \sin \theta (\hat{n} \times \hat{u}).
\]

or

\[
R(\hat{n}, \theta)\hat{u} = \hat{n}(\hat{n} \cdot \hat{u}) + \cos \theta [(\hat{n} - \hat{n}(\hat{n} \cdot \hat{u})] + \sin \theta (\hat{n} \times \hat{u})
\]

\[
R(\hat{n}, \theta)\hat{u} = (I - \cos \theta) \hat{n}(\hat{n} \cdot \hat{u}) + \sin \theta (\hat{n} \times \hat{u}) + \cos \theta \hat{u}
\]

**specifies action of** \( R(\hat{n}, \theta) \) **on arbitrary vector** \( \hat{u} \).

Now look at rotations about the \( \hat{x}, \hat{y}, \hat{z} \) axes. It's easy to write out the matrices for these (active convention as always).

\[
R(\hat{x}, \theta) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & -\sin \theta \\
0 & \sin \theta & \cos \theta
\end{pmatrix},
R(\hat{y}, \theta) = \begin{pmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{pmatrix}
\]

\[
R(\hat{z}, \theta) = \begin{pmatrix}
\cos \theta & -\sin \theta & 0 \\
\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}
\]
Now let's consider infinitesimal rotations (small $\theta$) about an axis $\hat{n}$.

Expand in Taylor series,

$$ R(\hat{n}, \theta) = I + \theta A + \ldots $$

Through 1st order.

where $I = R(\hat{n}, 0)$

$$ A = \left. \frac{d}{d\theta} R(\hat{n}, \theta) \right|_{\theta = 0} $$

A obviously depends on $\hat{n}$.

Use $R^T R = I$, $(I + \theta A^T + \ldots)(I + \theta A) = I + \theta (A^T + A) + \ldots = I$.

Thus $A^T + A = 0$, or $A$ is antisymmetric.

Convenient parameterization for an antisymmetric matrix,

$$ A = \begin{pmatrix} 0 & a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} = \sum_{i=1}^{3} a_i \hat{g}_i $$

where $\hat{g}_i$, $i=1,2,3$ are a "vector" of matrices,

$$ \hat{g}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \hat{g}_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \hat{g}_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} $$

Write this as $A = \hat{a} \cdot \hat{g}$, but you must remember that $\hat{a}$ is a vector of numbers, while $\hat{g}$ is a "vector" of matrices. Thus for $\theta \ll 1$ we have

$$ R(\hat{n}, \theta) = I + \theta \hat{a} \cdot \hat{g} \quad (\theta \ll 1) $$

for some vector $\hat{a}$. What is relation betw. $\hat{a}$ and $\hat{n}$?
Easy to see from a picture that
\[
R(\hat{n}, \theta) \vec{u} = \vec{u} + \theta \hat{n} \times \vec{u} \quad (\theta \ll 1)
\]
or you can make the approx. \( \cos \theta \approx 1, \sin \theta \approx \theta \) in the general formula for \( R(\hat{n}, \theta) \vec{u} \) worked out above.

But,
\[
\begin{pmatrix}
0 & -a_3 & a_2 \\
 a_3 & 0 & -a_1 \\
-a_2 & a_1 & 0
\end{pmatrix}
\begin{pmatrix}
u_1 \\
u_2 \\
u_3
\end{pmatrix} = \begin{pmatrix}
a_2 u_3 - a_3 u_2 \\
a_3 u_1 - a_1 u_3 \\
a_1 u_2 - a_2 u_1
\end{pmatrix} = \hat{a} \times \vec{u}.
\]

This is a useful property of \( \hat{a} \) matrices,
\[
(\hat{a} \cdot \vec{u}) \vec{u} = \hat{a} \times \vec{u} \quad \text{(any} \hat{a}\text{)}.
\]

It gives another notation for the cross product.

Thus for a near-identity (infinitesimal) rotation, we have
\[
R(\hat{n}, \theta) \vec{u} = \vec{u} + \theta \hat{n} \times \vec{u}
\]
\[
R(\hat{n}, \theta) \vec{u} = [I + \theta (\hat{a} \cdot \vec{u})] \vec{u} = \vec{u} + \theta \hat{a} \times \vec{u}.
\]

Thus for the \( \hat{a} \) that occurs in the infinitesimal rotation, we have \( \hat{a} = \hat{n} \), or...
\[ R(\hat{n}, \theta) = I + \theta \hat{n} \cdot \hat{\jmath} \quad (\theta \ll 1) \]

Recall for \( \mathfrak{so}(2) \) we had \( R(\theta) = I + \theta \hat{\jmath} \quad (\theta \ll 1) \). Should have called that \( I \)

Some useful properties of the \( \hat{\jmath} \) matrices:

\[
(\hat{\partial}_i)_{jk} = -\varepsilon_{ijk} \\
(\hat{\partial}_i \hat{\partial}_j)_{kl} = \delta_{ik} \delta_{jl} - \delta_{ij} \delta_{kl}
\]

From this you can derive the commutation relations,

\[
[\hat{\partial}_i, \hat{\partial}_j] = \varepsilon_{ijk} \hat{\jmath}^k
\]

Will explain importance of this soon. This can be written in another form,

\[
[\hat{\partial}_i \hat{\partial}_j, \hat{\jmath}^k] = (\hat{\partial}_i \times \hat{\jmath}^k) \cdot \hat{\jmath}^j.
\]

(The commutator of 2 antisymmetric matrices is antisymmetric.)

Finite rotations can be built up out of infinitesimal ones. Proceed as we did with \( \mathfrak{so}(2) \).

\[ R(\hat{n}, \theta + \phi) = R(\hat{n}, \theta) R(\hat{n}, \phi) = R(\hat{n}, \phi) R(\hat{n}, \theta). \]

Apply \( \frac{d}{d\phi} \bigg|_{\phi=0} \), use \( \frac{d}{d\phi} R(\hat{n}, \phi) \bigg|_{\phi=0} = \hat{n} \cdot \hat{\jmath} \). We get,

\[
\frac{d}{d\theta} R(\hat{n}, \theta) = (\hat{n} \cdot \hat{\jmath}) R(\hat{n}, \theta). \quad \text{diff. eqn, init cond.} \quad R(\hat{n}, 0) = I
\]

Solve,

\[ R(\hat{n}, \theta) = e^{\theta \hat{n} \cdot \hat{\jmath}} \]
Matrices $\vec{f}$ are considered the generators of the defining rep. of SO(3).

Now let $A = \hat{a} \cdot \vec{f}$ be an antisymmetric matrix. Let $R_0 \in SO(3)$ be a rotation. Consider

$$R_0 A R_0^T = R_0 (\hat{a} \cdot \vec{f}) R_0^T.$$ 

This is antisymmetric, $(R_0 A R_0^T)^T = R_0 A^T R_0^T = -R_0 A R_0^T$, so it must have the form $\vec{T} \cdot \vec{f}$ for some vector $\vec{T}$. What is the relation between $\vec{T}$ and $\vec{a}$? Answer is $\vec{T} = R_0 \vec{a}$, or...

$$R_0 (\hat{a} \cdot \vec{f}) R_0^T = (R_0 \vec{a}) \cdot \vec{f}. \quad \text{(Covariant formula)}.$$ 

Give it a name, call it adjoint formula.

To prove it, start with the fact that the cross product transforms as a vector under proper rotations, that is,

$$R_0 (\vec{a} \times \vec{u}) = (R_0 \vec{a}) \times (R_0 \vec{u}) = \vec{T} \times \vec{w} \quad \text{where} \quad \vec{T} = R_0 \vec{a}, \vec{w} = R_0 \vec{u}.$$ 

But, write the cross products in terms of the $\vec{f}$ matrices,

$$R_0 (\hat{a} \cdot \vec{f}) \vec{u} = (\vec{T} \cdot \vec{f}) \vec{w} = (\vec{T} \cdot \vec{f}) R_0 \vec{u} = [(R_0 \vec{a}) \cdot \vec{f}] R_0 \vec{u}$$

or since $\vec{u}$ is arbitrary,

$$R_0 (\hat{a} \cdot \vec{f}) = [(R_0 \vec{a}) \cdot \vec{f}] R_0$$

or

$$R_0 (\hat{a} \cdot \vec{f}) R_0^T = (R_0 \vec{a}) \cdot \vec{f}. \quad \text{QED formula above.}$$
Now consider...

\[ \begin{align*}
R_0 \ R(\hat{n}, \theta) R_0^T &= R_0 \ e^{\theta (\hat{n}, \vec{\gamma})} R_0^T \\
&= R_0 \left[ I + \theta (\hat{n}, \vec{\gamma}) + \frac{\theta^2}{2!} (\hat{n}, \vec{\gamma})^2 + \ldots \right] R_0^T \\
\text{note,} \quad R_0 (\hat{n}, \vec{\gamma})^2 R_0^T &= R_0 (\hat{n}, \vec{\gamma}) R_0^T R_0 (\hat{n}, \vec{\gamma}) R_0^T \\
&= \left[ (R_0 \hat{n}, \vec{\gamma}) \right]^2 \quad \text{similarly other powers.} \\
\Rightarrow \quad I + \theta \left[ (R_0 \hat{n}, \vec{\gamma}) \right] + \frac{\theta^2}{2!} \left[ (R_0 \hat{n}, \vec{\gamma}) \right]^2 + \ldots \\
&= e^{\theta \left( R_0 \hat{n}, \vec{\gamma} \right)} = R(R_0 \hat{n}, \theta).
\end{align*} \]

Or, \[ R_0 \ R(\hat{n}, \theta) R_0^T = R(R_0 \hat{n}, \theta) \quad \text{exponentiated version of adjoint formula.} \]

So this is the proof of the fact that when you conjugate a rotation by another rotation, it rotates the axis but doesn't change the angle.

Now let's consider the fact that \( SO(3) \) is non-abelian. In the case of rotations about a fixed axis, we have a simple multiplication law,

\[ R(\hat{n}, \theta_1) \ R(\hat{n}, \theta_2) = R(\hat{n}, \theta_1 + \theta_2), \]

because such rotations commute. (They belong to an \( SO(2) \) subgroup.) But rotations about different axes don't commute. So let's look at 2 axes \( \hat{n}_1, \hat{n}_2 \), with \( \hat{n}_1 \neq \hat{n}_2 \). Then there is no simple formula for \[ R(\hat{n}_1, \theta_1) \ R(\hat{n}_2, \theta_2). \]
However, the result is certainly a rotation, so we can write
\[ R(\hat{n}_1, \theta_1) R(\hat{n}_2, \theta_2) = R(\hat{n}_3, \theta_3) \]
for some \( \hat{n}_3, \theta_3 \) which are functions of \( \hat{n}_1, \theta_1 \) and \( \hat{n}_2, \theta_2 \). But the functions are not simple, due to the noncommutativity.

However, there is some simplification if the angles \( \theta_1, \theta_2 \) are small. To begin analyzing this question, let \( R_1, R_2 \) be rotations, and let
\[ C = R_1 R_2 R_1^{-1} R_2^{-1}, \]
which is also a rotation. \( C \) is a measure of the noncommutativity of \( R_1, R_2 \), since \( C = I \) if \( R_1 \) and \( R_2 \) should commute. Now let \( R_1 = e^{\hat{a}_1 \cdot \vec{f}}, \ R_2 = e^{\hat{a}_2 \cdot \vec{f}} \) where \( \hat{a}_1 = \hat{n}_1 \theta_1, \ \hat{a}_2 = \hat{n}_2 \theta_2 \), and expand in powers of \( \hat{a} \):
\[
C = \left[ I + \hat{a}_1 \cdot \vec{f} + \frac{1}{2} (\hat{a}_1 \cdot \vec{f})^2 + \ldots \right] \\
\times \left[ I + \hat{a}_2 \cdot \vec{f} + \frac{1}{2} (\hat{a}_2 \cdot \vec{f})^2 + \ldots \right] \\
\times \left[ I - \hat{a}_1 \cdot \vec{f} + \frac{1}{2} (\hat{a}_1 \cdot \vec{f})^2 + \ldots \right] \\
\times \left[ I - \hat{a}_2 \cdot \vec{f} + \frac{1}{2} (\hat{a}_2 \cdot \vec{f})^2 + \ldots \right].
\]
Expand this out, you find that all terms 1st order in the \( \hat{a} \)'s vanish, and you must go to 2nd order to get a correction from the identity. Thus, you find,
\[
C = I + [\hat{a}_1 \cdot \vec{f}, \hat{a}_2 \cdot \vec{f}] + \theta(\alpha^2) = I + (\hat{a}_1 \times \hat{a}_2) \cdot \vec{f} + \theta(\alpha^2).
\]
For example, a rotation

\[ C = R_x(\theta_1) R_y(\theta_2) R_x(-\theta_1) R_y(-\theta_2) \]

is not the identity, but is a small rotation about the z-axis (assuming \( \theta_1, \theta_2 \) are small).

**Moral:** The commutativity of near identity rotations is expressed in terms of the commutator

\[ [\vec{a}_1 \cdot \hat{y}, \vec{a}_2 \cdot \hat{y}] = (\vec{a}_1 \times \vec{a}_2) \cdot \hat{y} \]

of the generators.