Begin with HW problem #2, concerning \( C_{3v} \). Covering operations of \( \text{NH}_3 \) molecule.

\[
E = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_3 = \sigma_U = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
R_1 = R_2(2\pi/3) = \begin{pmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_4 = \sigma_V R_1 = \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
R_2 = R_2(4\pi/3) = \begin{pmatrix} -1/2 & \sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad R_5 = \sigma_U R_2 = \begin{pmatrix} 1/2 & -\sqrt{3}/2 & 0 \\ -\sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

These come from applying formula,

\[
R e_i = \sum_j e_j R_{ji}, \quad \text{or} \quad R_{ij} = (e_i, R e_j)
\]

where \((e_1, e_2, e_3) = (\hat{x}, \hat{y}, \hat{z})\).

\( \tilde{R} \) with \( \tilde{R} \) means induced transform. 

\[
(\tilde{R} \psi)(\vec{r}) = \psi(\vec{R}^{-1} \vec{r})
\]

Basis, \((\psi_1, \psi_2, \psi_3, \psi_4, \psi_5, \psi_6) = (x^2, y^2, z^2, xy, yz, zx)\), a 6-dimensional space, invariant under \( C_{3v} \).
So, expand the action of $\tilde{R}$ on the basis functions,

$$(\tilde{R} \psi_i)(\tilde{\Phi}) = \psi_i (R^{-1} \tilde{\Phi}) = \sum_j \psi_j (\tilde{\Phi}) \tilde{R}_{ij}$$

You find a set of 6x6 matrices $\tilde{R}_{ij}$, of which 3 are the following:

$$R_1 = \begin{pmatrix}
\frac{1}{4} & \frac{3}{4} & 0 \\
\frac{3}{4} & \frac{1}{4} & 0 \\
0 & 0 & 1 \\
-\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \\
o & 0 & 0 \\
o & 0 & 0 \\
\end{pmatrix}$$

$$R_4 = \begin{pmatrix}
\frac{1}{4} & \frac{3}{4} & 0 \\
\frac{3}{4} & \frac{1}{4} & 0 \\
0 & 0 & 1 \\
\frac{\sqrt{3}}{2} & -\frac{\sqrt{3}}{2} & 0 \\
o & 0 & 0 \\
o & 0 & 0 \\
\end{pmatrix}$$

$$R_5 = \begin{pmatrix}
\frac{1}{4} & \frac{3}{4} & 0 \\
\frac{3}{4} & \frac{1}{4} & 0 \\
0 & 0 & 1 \\
-\frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} & 0 \\
o & 0 & 0 \\
o & 0 & 0 \\
\end{pmatrix}$$
These were from the homework, the other 3 matrices are easy.

\[ m_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \] (obvious).

\[ R_3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \] (easy, because \( R_3 \) takes \( x \) to \(-x\), leaves \( y, z \) alone).

\[ R_2 = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 & -\frac{\sqrt{3}}{2} & 0 & 0 \\ \frac{\sqrt{3}}{2} & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & -\sqrt{3} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \\ 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{pmatrix} \] (use \( \tilde{R}_2 = \tilde{R}_3 \tilde{R}_5 \)).

We have here 2 examples of what are called representations of a group: in this case, representations of \( G_{3v} \) by \( 3 \times 3 \) and \( 6 \times 6 \) matrices. The \( 3 \times 3 \) matrices are orthogonal, but the \( 6 \times 6 \) matrices are not.

In general, a representation of a group \( G \) is an association \[ a \mapsto M(a) \] of group elements \( a \in G \) with matrices \( M(a) \), such that \[ M(a) M(b) = M(ab) . \] (\( M = "matrix" \)) (The matrix multiplication reproduces the group multiplication law.)
(Of course, a matrix stands for a linear operator in some basis. We may prefer to define a representation as an association between group elements and linear operators that map some vector space into itself. This is a slightly more abstract point of view.) Also “carrier space”

The vector space in question is called the representation space (by the book). See prob. 1 of HW #3.

A representation is the same as a group action on a vector space, with the additional requirement that the action be linear.

As we see, a given group may (usually does) have more than one representation.

Simple consequences of this definition are

\[ M(e) = E \quad \text{(identity matrix).} \]

\[ M(a^{-1}) = M(a)^{-1}. \quad \text{(thus, the matrices } M(a) \text{ are invertible)} \]

Every group has a trivial representation in terms of 1x1 matrices,

\[ M(a) = (1), \text{ for all } a \in G. \]

For the group $C_{3v}$, here is another representation by 1x1 matrices:

\[ M(E) = M(R_1) = M(R_2) = (1) \]

\[ M(R_3) = M(R_4) = M(R_5) = (-1) \]

(Any group)

The trivial rep. and this 1x1 rep. of $C_{3v}$ are not faithful.

In general, a faithful representation is a one-to-one association between group elements and matrices (that is, all the matrices are distinct).

A faithful representation is an isomorphism between the abstract group and a group of matrices. If the representation is not faithful, then we only have a homomorphism.
From here for a while we will work on the problem of finding all possible representations of a group. Outline of the steps:

1. Define equivalent, inequivalent reps.

2. Maschke’s Theorem: all reps of a finite group are equivalent to unitary reps.

3. Invariant subspaces and reducibility (decomposing a given rep. into smaller reps.)

4. Define irreducible rep. (one that cannot be decomposed into smaller reps.)

5. Schur’s lemmas #1 and #2 (criterion for irreducibility)

6. Grand Orthogonality Theorem (miraculous properties of unitary irreps.)
An important problem in physics is, given a group (the multiplication law), find all distinct representations of that group.

But before we do that, we must define what we mean by distinct. Certainly, we will consider two representations as distinct if they have different dimensionality. What if they have the same dimensionality?

A trivial observation is that if we have a representation \( \mathbf{a} \mapsto \mathbf{M}(\mathbf{a}) \) by (say) \( nxn \) matrices, we can get another \( nxn \) representation just by changing basis. If \( \mathbf{S} \) is the \( nxn \) matrix that brings about the change of basis (maps the old basis vectors into the new ones), then

\[
\mathbf{M}'(\mathbf{a}) = \mathbf{S} \mathbf{M}(\mathbf{a}) \mathbf{S}^{-1}
\]

where \( \mathbf{M}'(\mathbf{a}) \) are the new matrices. Note that \( \mathbf{S}^{-1} \) always exists. (because it's a change of basis).

If this is the case, then the representations \( \mathbf{M}(\mathbf{a}) \) and \( \mathbf{M}'(\mathbf{a}) \) are said to be equivalent. Equivalent representations are considered distinct only in a trivial sense.

If we have two representations of the same dimension, \( \mathbf{M}(\mathbf{a}) \) and \( \mathbf{M}'(\mathbf{a}) \), such that they are not related by \( \mathbf{M}'(\mathbf{a}) = \mathbf{S} \mathbf{M}(\mathbf{a}) \mathbf{S}^{-1} \) for any \( \mathbf{S} \), then the representations are called inequivalent. They are also called inequivalent if the dimensions are different.

So clearly, in looking for representations of a group, we should only look for inequivalent representations.
sometimes we can take a representation of a given size and extract a smaller representation from it. For example, in the 3x3 rep. of C\(_3\)v above, all 6 matrices are block-diagonal, that is they have the form,

\[
\begin{pmatrix}
X & X & 0 \\
X & X & 0 \\
0 & 0 & X
\end{pmatrix}
\]

\[i = 1, 2, \ldots, 6\]

\[X = \text{anything}\]

\[0 = \text{zero}.\]

This means that the 2x2 upper block forms a 2x2 rep. of C\(_3\)v, and the 1x1 lower block does also. (In fact, the 1x1 lower block is the trivial rep.) The 3x3 rep. decomposes into smaller reps.

Of course, if we changed basis, the 3x3 rep. of C\(_3\)v would no longer be block diagonal. So it may be hard by looking at some matrices to see if they can be decomposed or reduced (as we say) to smaller matrices. The method we can use is block diagonal.

For example, the 6x6 rep. of C\(_3\)v can also be reduced to matrices of smaller dimensionality, but it is not obvious by looking at the matrices. (It is not obvious that a change of basis will block diagonalize these 6x6 matrices.)

The geometrical criterion for being able to extract a rep. of lower dimensionality from a given rep. is the existence of an invariant subspace. Given a representation \(\rho: G \rightarrow \text{GL}(V)\) of a group \(G\) by \(n \times n\) matrices, we say that \(\rho\) is
The following discussion is simplified if we restrict consideration to unitary representations, that is, representations in which the matrices \( M(g) \) are unitary. For finite groups there is no loss of generality in this, since it turns out that any representation of a finite group is equivalent (by some change of basis) to a unitary rep. The book calls this Maschke's theorem.

Here is the proof: There are 3 major steps. Let \( g \mapsto M(g) \) be a representation of a group of finite order by finite dimensional matrices. Note, \( M(g^{-1}) = M(g)^{-1} \), but \( M(g) \) is not assumed to be unitary so \( M(g)^+ \neq M(g) \).

**Step 1.** Define \( S = \sum_{g \in G} M(g)^+ M(g) \). (sum over all group elements)

Observe that \( S \) is Hermitian, \( S = S^+ \), hence it can be diagonalized by a unitary matrix and its eigenvalues are all real,

\[
S = A \begin{pmatrix}
\lambda_1 & \cdots & 0 \\
0 & \cdots & \lambda_n
\end{pmatrix} A^+ , \quad A \text{ unitary,} \quad \text{all } \lambda_i \text{ real.}
\]

Note also that \( S \) is positive definite, which means that all eigenvalues are positive. Thus we can define the square root of \( S \), call it \( K \),

\[
K = \sqrt{S} = A \begin{pmatrix}
\sqrt{\lambda_1} & \cdots & 0 \\
0 & \cdots & \sqrt{\lambda_n}
\end{pmatrix} A^+
\]

so that \( K \) is also Hermitian and positive definite, and \( K^2 = S \).
Aside on positive definite matrices. A Hermitian matrix $H$ is positive definite if $(x, Hx) \geq 0$ for all vectors $x$, and $(x, Hx) = 0$ iff $x = 0$. This is the usual definition, but it's equivalent to saying that all the eigenvalues are positive, because if you compute $(x, Hx)$ in the basis where $H$ is diagonal, you get

$$(x, Hx) = \sum_{i=1}^{n} \lambda_i |x_i|^2$$

where $x_i$ = components of $x$ \quad $\lambda_i > 0$, eigenvalues of $H$.

which clearly is $\geq 0$ and in fact $= 0$ iff every $x_i = 0$.

The matrix $S$ above is pos. def. because

$$(x, SX) = \sum_{g \in G} (x, M(g)^+ M(g) x)$$

$$\Rightarrow = (M(g)x, M(g)x) = |M(g)x|^2 \geq 0. \quad (\text{all terms pos.})$$

$m(g)$ is invertible, so $M(g)x = 0$ iff $x = 0$, so $(x, SX) = 0$ iff $x = 0$.

**Step 2.** Work out what happens when you commute one of the $M$'s through $S$. Let $a \in G$, consider,

$$SM(a) = \sum_{g \in G} M(g)^+ M(g) M(a)$$

$$= \sum_{g \in G} M(g)^+ M(ga).$$

Let $b = ga$. Think of $a$ as fixed. When $g$ runs through the group (in the sum), $b$ does also (b also runs through the group). This is the rearrangement theorem. So summing $g \in G$ is the...
same as summing \( b \in G \). Also, \( g \in G \) so,

\[
S M(a) = \sum_{b \in G} M(ba^{-1})^+ M(b) \quad \text{(change of variable of summation)}
\]

\[
= \sum_{b \in G} [M(b)M(a^{-1})]^+ M(b)
\]

\[
= \sum_{b \in G} M(a^{-1})^+ M(b)^+ M(b)
\]

\[
= M(a^{-1})^+ S.
\]

Summary: \( S M(a) = M(a^{-1})^+ S \)

or \( K^2 M(a) = K \cdot M(a^{-1})^+ K^2 \).

**Step 3.** Define

\( U(a) = K M(a) K^{-1} \),

\[ U(a)^+ = K^{-1} M(a)^+ K \quad \text{since } K=K^+. \]

\[ U(a)^+ U(a) = K^{-1} M(a)^+ K^2 M(a) K^{-1} \]

\[ = K^{-1} M(a)^+ M(a^{-1})^+ K^2 K^{-1} \quad \text{(identity above)} \]

\[ = K^{-1} [M(a^{-1}) M(a)]^+ K \]

\[ = K^{-1} E K = K^{-1} K = E. \]

So, \( U(a) \) is unitary, all \( a \in G \).

\text{Q.E.D.}

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End of Maschke's Thm.
Remarks on Maschke's thm: The proof of this theorem doesn't necessarily work if the group is of infinite order, because the sum $\sum f$ might not converge. In particular, a different approach is needed for Lie groups like $SO(3)$ and $SU(2)$.

Lie groups can be divided into 2 classes, the compact and noncompact. The compact groups work very much like finite groups, for example, all the reps of a compact group are equivalent to a unitary rep. This applies for example to $SO(3)$ and $SU(2)$, and similarly for noncompact groups.

For the noncompact groups (like the Lorentz group) this is no longer true, and one must work with nonunitary representations. One sees this, for example, in the Dirac equation.

Finite and compact groups are the easiest cases, here there is no loss of generality in assuming that all reps are unitary. We will (probably) not deal with noncompact groups in this course.
Henceforth we may assume that all representations we deal with are **unitary**, so that

\[ M(g^{-1}) = M(g)^{-1} = M(g)^+ \]

Now go back to problem of extracting smaller dimensional reps. from a given rep. This revolves around the issue of **invariant subspaces**.

Let \( g \mapsto T(g) \) be a representation of a group by means of unitary operators \( T(g) \) that act on some vector space \( V \). The difference between \( T(g) \) and \( M(g) \) is that \( T(g) \) are the operators and \( M(g) \) are the matrices that represent these operators in some basis. We choose an orthonormal basis so that the matrices \( M(g) \) will be unitary.

Let \( W \) be the set of unitary operators

Suppose \( \{ T(g) \} \) possesses an **invariant subspace** \( W \), that is, for some subspace \( W \) of \( V \), suppose that if \( x \) is a vector in \( W \), then so is \( T(g)x \) for all \( g \in G \). (In the 3x3 repn of \( C_3 \) the \( x-y \) plane and the \( z \)-axis are 2- and 1-dimensional invariant subspaces, respectively.)

Let \( \dim V = n \) and \( \dim W = m \), with \( 0 \leq m \leq n \).

Choose \( m \) orthonormal basis vectors \( \{ e_1, \ldots, e_m \} \) that span \( W \) and then \( n-m \) more vectors orthogonal to these, \( \{ e_{m+1}, \ldots, e_n \} \) that altogether form an orthonormal basis in \( V \). The \( n-m \) basis vectors \( \{ e_{m+1}, \ldots, e_n \} \) span the space orthogonal to \( W \), call it \( W^\perp \), so that

\[ W \oplus W^\perp = V. \]
Now look at the matrices $M(g)$ that represent $T(g)$ in this basis. Naturally, you have to break these matrices up into their $n = m + (n-m)$ parts like this,

$$
M(g) = \begin{pmatrix}
M_{1\,1}(g) & M_{1\,2}(g) \\
M_{2\,1}(g) & M_{2\,2}(g)
\end{pmatrix}
$$

Also, if $x$ is any vector in $V$, break it up,

$$
x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}
$$

Now if $x$ happens to lie in $W$, then $x_2 = 0$, or $x = \begin{pmatrix} x_1 \\ 0 \end{pmatrix}$. And since $W$ is an invariant subspace, if we let $M(g)$ act on this, we must get another vector of the same form:

$$
\begin{pmatrix}
M_{1\,1}(g) & M_{1\,2}(g) \\
M_{2\,1}(g) & M_{2\,2}(g)
\end{pmatrix}
\begin{pmatrix} x_1 \\ 0 \end{pmatrix} =
\begin{pmatrix}
M_{1\,1}(g) x_1 \\
M_{2\,1}(g) x_1
\end{pmatrix}
$$

But if $M_{2\,1}(g)x_1 = 0$ for all $x_1$, it must be that $M_{2\,1}(g) = 0$. Thus, the lower left corner of $M(g)$ vanishes, for all $g$. 
Now use the fact that $M(g)$ is unitary, so that...

$$M(g) = \begin{pmatrix} M_{11}(g) & M_{12}(g) \\ 0 & M_{22}(g) \end{pmatrix} \quad \text{for all } g \in G.$$  $$\frac{M(g)}{g} = \begin{pmatrix} M_{11}(g)^* & 0 \\ 0 & M_{22}(g)^* \end{pmatrix} = M(g^{-1}).$$

But this means that $M_{12}(g)^* = 0$ or $M_{12}(g) = 0$. Thus the upper right corner of $M(g)$ must also vanish.

Thus if $\{T(g)\}$ has an invariant subspace, then in this basis $M(g)$ is block-diagonal,

$$M(g) = \begin{pmatrix} M_{11}(g) & 0 \\ 0 & M_{22}(g) \end{pmatrix}$$

and the original rep. $M(g)$ has been reduced to 2 smaller reps. of dimension $m$ and $n-m$.

This is not useful unless $1 \leq m < n$. The cases $m=0$ ($W=\{0\}$) or $m=n$ ($W=V$) are trivial; these are trivial invariant subspaces, and don't give rise to new representations (since either $M_{11}=M$ or $M_{22}=M$, and the other is 0-dimensional).
Thus we say that the representation \( g \mapsto T(g) \) is reducible if the set of operators \( \{ T(g) \} \) possesses a nontrivial, invariant subspace. If \( \{ T(g) \} \) has no invariant subspace apart from the trivial examples \( W = \{ 0 \} \) and \( W = V \), then we say that the rep. \( g \mapsto T(g) \) is irreducible. Irreducible representations cannot be decomposed into representations of smaller size.

Finally, to classify all the representations of a group, we look for all inequivalent, irreducible representations.

If a rep. is reducible as above, the blocks \( M_{11}(g) \) and \( M_{22}(g) \) may themselves be reducible. If they are, then they can be block diagonalized by further change of basis, producing smaller and smaller matrices, until nothing but irreducible representations remain. Thus, \( M(g) \) looks like,

\[
M(g) = \\
\]

where each block has some definite size and is irreducible.
some of the irreducible blocks on the diagonal may be of the same size. If they are, they may or may not be equivalent. If they are equivalent, then we have repetitions of an irreducible representation (henceforth irrepp) in the decomposition of the original, reducible rep. M(g).

By a change of basis, all copies of a given irrepp can be brought into the same set of matrices. That is, for each irreducible representation, let us choose a standard basis, producing a standard set of matrices, call them $M^{(r)}(g)$, where $r$ is a label of the irrepp. Let $d_r = \dim M^{(r)}$, so that $\{M^{(r)}(g)\}$ is a set of $d_r \times d_r$ matrices.

Then the result of the definitions and theorems above is that by a change of basis, an arbitrary, unitary rep. $g \mapsto M(g)$ of a group can be block diagonalized, producing some number of repetitions of each irrepp on the diagonal, that is some number of copies of $M^{(r)}(g)$.

Thus, the problem of classifying the reps of a group boils down to this: Find all inequivalent, irreducible representations (because an arbitrary rep. is built up out of these, by making block diagonal matrices and then changing basis).

Questions: Given G, how many irreps are there? Do they form continuous or discrete families? How do we find them explicitly? Etc. Etc. (next project.)