Continue today with some more abstract group theory.

Last time quickly discussed some consequences of Lagrange's theorem. Recall, this theorem says that the cosets of a subgroup $H \leq G$ (either right or left cosets) divide $G$ up into disjoint subsets of $\#(H)$ elements each. Hence $\#H$ is a divisor of $\#G$.

Now define the order of an element $g \in G$ as the smallest $n$ such that $g^n = e$.

(Aside: Consider generating elements by taking powers of $g$. Suppose you have created the list, $g, g^2, \ldots, g^n$ for $n \geq 1$, and suppose this list contains $n$ distinct elements. Then you can show that either $g^{n+1}$ is distinct from all the previous elements, or else $g^{n+1} = g^0$, in which case $g^n = e$. In other words, when the list repeats, it (the repetition) starts from the beginning, not in the middle. Now, assuming $G$ has finite order, then the order of each group element must also be finite, because you can't go on generating distinct elements forever, in fact, $\text{order}(g) \leq \text{order}(G) = \#(G)$. End aside.)

So every group element $g$ generates a cyclic subgroup of order $n$, where the subgroup is $(e, g, g^2, \ldots, g^{n-1})$ and $g^n = e$ ($n = \text{order}(g)$). But by Lagrange's theorem, $n$ must divide $\#(G)$.

So for example in $D_3$ (group of order 6) every group element must have an order that is $1, 2, 3,$ or $6$.
From this it follows that a group of prime order (like \#1=7) has only one possible structure: it is the cyclic group.

Previously we showed that there is only one (abstract) gp. of order \#1 and \#2. The case \#2 is a special case of the above, so we can start to fill in a table:

\[
\begin{array}{cc}
 n & \# \text{ of groups of order } n \\
1 & 1 \\
2 & 1 \\
3 & 1 \\
4 & ? \leftarrow \text{HW problem.} \\
5 & 1 \\
6 & ? \leftarrow \text{1st non-Abelian gp appears here, isomorphic to } D_3 \\
7 & 1
\end{array}
\]

So let's do the HW problem. How many distinct (non-isomorphic) groups of order 4 are there?

Well, the group has 4 elements of which one must be the identity, \( G = \{e, g, g^2, g^3\} \). The order of \( e \) is 1, and the order of the others must be \( \geq 1 \), that is, 2 or 4 (by Lagrange's thm.). Suppose one of the elements \( g \) has order 4. Then \( G = \{e, g, g^2, g^3\} \) is the cyclic group of order 4. Otherwise, suppose no element of \( G \) has order 4. Then \( g_1, g_2, g_3 \) (call them \( a, b, c \) now) must all have order 2, \( a^2 = b^2 = c^2 = e \).
So we get the multiplication table,

\[
\begin{array}{ccc}
  e & a & b & c \\
  a & e^2 & & \\
  b & & e^2 & \\
  c & & & e^2 \\
\end{array}
\]

Now consider product \( ab \). It must equal \( e, a, b \) or \( c \). But

\[
\begin{align*}
  ab &= e & \text{impossible since } a^2 &= a, \text{ not } b. \\
  ab &= a & \text{impossible since } b &= e \\
  ab &= b & \text{impossible since } a &= e
\end{align*}
\]

So \( ab \) must be \( c \). In this way we can fill in the rest of the table in a unique way. This group is called the \( 4 \)-group or \( 4 \)-ruppe.

Thus there are 2 abstract gaps of order 4 (both Abelian).

Another useful definition:

A \textbf{homomorphism} between groups \( G \) and \( K \) is a map \( f : G \to K \) such that \( f(a)f(b) = f(ab) \) for all \( a, b \in G \).

( Ignore the definition of homomorphism given on p. 15 of last week's notes. It was slightly wrong.)

This defn is just like that of an \textbf{isomorphism}, except the map is not required to be one-to-one and onto (it can be any map that preserves the multiplication law.)
Example of homomorphism: Let $D_3 = \text{group of proper covering operations of equilateral } \Delta$, as in last week's notes, see pp. 5, 6 for defn's of $R_1, R_2, \ldots, R_5$. Then the map,

\[
G = \begin{bmatrix} e \\ R_1 \\ R_2 \\ R_3 \\ R_4 \\ R_5 \end{bmatrix}
\xrightarrow{f} \begin{bmatrix} e \\ a \\ a \end{bmatrix} = K \quad \text{with } a^2 = e
\]

is a homomorphism. You can see this because the operations \([E, R_1, R_2]\) do not flip the $z$-axis, while \([R_3, R_4, R_5]\) do. Therefore the product of any two members of 2nd subset must be a member of the first \((a^2 = e)\), and likewise for $ae = ea = a$, $e^2 = e$.

[This is the beginning of some interesting facts about homomorphisms that we will skip for now.]

An important homomorphism in physics: You have heard that if you rotate an electron (a spin $\frac{1}{2}$ particle) by $360^\circ$, you do not get the original electron back, but rather it suffers a phase change of $-1$. You have to rotate by $720^\circ$ to restore the electron to its original state.

The group of electron rotations is $SU(2)$, the group of ordinary (3D) rotations is $SO(3)$. There is a 2-to-1 homomorphism between these,

\[
\begin{array}{ccc}
SU(2) & \longrightarrow & SO(3) \\
\end{array}
\]
such that

\[ 1 = U_\pi(2\pi) \quad \rightarrow \quad R_\pi(2\pi) = E \]

\[ -1 = U_\pi(4\pi) \quad \rightarrow \quad R_\pi(4\pi) \]

More generally,

\[ U_\pi(\theta) \rightarrow R_\pi(\theta) \quad \text{for any } 0 \leq \theta < 2\pi. \]

New defn.: Given group \( G \), \( a, b \in G \). Then \( a, b \) are said to be conjugate if there exists \( g \in G \) such that

\[ a = gbg^{-1}. \]

Note: if \( a \) conj to \( b \) and \( b \) conj to \( c \), then \( a \) conj to \( c \).

The interpretation of conjugate elements in physical applications is often that they represent "the same thing" as seen by "two different observers." Cf. analogy in matrix algebra. Let \( x, y \) = vectors and \( M \) = a matrix, and suppose \( y = Mx \). Now let \( T \) = a transformation (maybe a change of coordinates), and let \( x' = Tx, y' = Ty \).

Then

\[ y' = Ty = TMx = TMT^{-1}x' = M'x' \]

\( \text{[a conjugation note]} \)

if we write \( M' = TMT^{-1} \). If \( T \) represents a change of coordinates, then \( M, M' \) represent the same process as viewed from 2 coordinate systems.
Another defn. Let $G = \text{group}$, $a \in G$. Then the set of elements $\{ gag^{-1}, g \in G \}$ is called a conjugate class.

(It is the set of all elements in a group that are conjugate to $a$).

Every element in $G$ belongs to some conjugate class, and the conjugate classes are disjoint. Therefore the group $G$ breaks up into a set of conjugate classes.

Note that the identity $e$ is in a class by itself. Fe3

Note that for an Abelian group, every element commutes, is in a conjugate class by itself, since $a \cdot gag^{-1} = agg^{-1} = a$ for all $g \in G$.

Don't confuse conjugate classes with cosets. In both cases we have a way of breaking a group up into subsets, but the coset construction requires a subgroup $H \leq G$, while classes do not.

Also, the cosets are all of equal size = $\#(H)$, whereas conj. classes are generally of different sizes.

Example of $D_3 = \{ E, R_1, R_2, R_3, R_4, R_5 \}$. See p.7, 11/10/03 notes for group table.

Then you find the following conj. classes:

$\{ E \}, \{ R_1, R_2 \}, \{ R_3, R_4, R_5 \}.$
$R_3, R_4, R_5$ are all "alike" physically (rotations by $\pi$ in $xy$ plane).

Sim. $R_1, R_2$ are "alike" (rotations by $2\pi/3, 4\pi/3$ about $z$-axis).

---

**Study conjugate classes in $SO(3)$**.

1st, Some facts. Every proper rotation can be written $R(\hat{n}, \theta)$ for some axis and angle. This is geometrically pretty obvious. The axis $\hat{n}$ ranges over unit sphere. If we let $0 \leq \theta < 2\pi$ most rotations are counted twice, since

$$R(\hat{n}, \theta) = R(-\hat{n}, 2\pi - \theta)$$

Angles are measured w. right-hand rule. So you only need to let angle $\theta$ range between 0 and $\pi$; if $\hat{n}$ runs over whole sphere, it order to represent most rotations only once. (Angles $\theta = 0$ and $\theta = \pi$ are still special cases.)

The axis of a rotation is left invariant by the rotation,

$$R(\hat{n}, \theta) \hat{n} = \hat{n}$$

(geometrically obvious), $\hat{n}$ is an eigenvector of $R(\hat{n}, \theta)$ with eigenvalue 1.

As for other vectors, let $\hat{u}, \hat{v}$ be orthog. unit vectors spanning plane $\perp$ to $\hat{n}$, so that $(\hat{u}, \hat{v}, \hat{n})$ forms a right-handed triad, $\hat{u} \times \hat{v} = \hat{n}$. 

---
Then

\[ R(\hat{n}, \theta) \hat{u} = \cos \theta \hat{u} + \sin \theta \hat{v} \]
\[ R(\hat{n}, \theta) \hat{v} = -\sin \theta \hat{u} + \cos \theta \hat{v} \]

again geometrically obvious.

Now let \( R_0 \) = some fixed rotation, and let

\[ R' = R_0 R(\hat{n}, \theta) R_0^{-1}. \]

(a conjugation).

What is axis of \( R' \)? Claim it is \( \hat{n}' = R_0 \hat{n} \). 

Proof:

\[ R' \hat{n}' = (R_0 R(\hat{n}, \theta) R_0^{-1})(R_0 \hat{n}) \]
\[ = R_0 R(\hat{n}, \theta) \hat{n} \]
\[ = R_0 \hat{n} \]
\[ = \hat{n}'. \]

So \( R' = R(\hat{n}', \theta') \) for some angle \( \theta' \). What is \( \theta' \)?
Let \( \hat{u}' = R_0 \hat{u} \) and \( \hat{v}' = R_0 \hat{v} \) so that \( (\hat{u}', \hat{v}', \hat{n}') \) is a new orthonormal triad.

Then \( R' \hat{u}' = (R_0 R(\hat{n}, \theta) R_0^{-1}) (R_0 \hat{u}) \)
\[ = R_0 R(\hat{n}, \theta) \hat{u} \]
\[ = R_0 \left[ \cos \theta \hat{u} + \sin \theta \hat{v} \right] \]
\[ = \cos \theta \hat{u}' + \sin \theta \hat{v}' \]

Similarly, \( R' \hat{v}' = -\sin \theta \hat{u}' + \cos \theta \hat{v}' \).

So \( R' \) is a rotation about axis \( \hat{n}' \) by angle \( \theta \), \( \theta' = \theta \).

Altogether,
\[ R_0 R(\hat{n}, \theta) R_0' = R(R_0 \hat{n}, \theta) \]
(when you conjugate a proper rotation by another proper rotation, the axis rotates and the angle doesn't change.)

Thus: The conjugate classes in \( SO(3) \) are rotations of a fixed angle \( \theta \) \((0 \leq \theta \leq \pi)\) but arbitrary axis.

This means that for subgroups of \( SO(3) \), like \( D_3 \), rotations in a given conjugate class must have the same angle \( 0 \leq \theta \leq \pi \). However, if they have the same angle, it does not necessarily follow that they belong to the same conjugate class. It all depends on whether there is another rotation that maps (within group).
the axes of the 2 rotations into one another. Can see this in example of $D_3$.

Another concept from abstract group theory: How to build new groups from old ones. Simplest way is the direct product group. First, abstract definition.

**Def.**

Let $G, H$ be groups. Let $K$ = set of elements of form $(g, h)$, for $g \in G, h \in H$. Define multiplication of elements of $K$ by

$$(g_1, h_1)(g_2, h_2) = (g_1g_2, h_1h_2).$$

Then $K$ is a group, the direct product group, written

$K = G \times H$.

Note also, $(g, h)^{-1} = (g^{-1}, h^{-1})$,

$e_K = (e_G, e_H)$ (identity)

$|K| = |G| \cdot |H|$.  

Notice, $K$ contains subgroups isomorphic to $G, H$:

Subgroup \{ $(g, e_H), g \in G$ \} $\cong G$

and \{ $(e_G, h), h \in H$ \} $\cong H$

$\cong$ means, isomorphic to.

Notice also, any element of one subgroup commutes with any element of the other,

$$(g, e_H) \cdot (e_G, h) = (g, h) = (e_G, h) \cdot (g, e_H)$$

and every element in $K$ can be written as such a product.
Book's definition turns this around, says $K$ is a direct product of $G \times H$ if $G, H$ are subgroups of $K$ such that every element of $G$ commutes with every element of $H$ and such that every element of $G$ can be written uniquely as a product of an element of $G$ times an element of $H$.

Where direct product groups tend to occur in physics: When $G$ and $H$ "act on different spaces," or for other reasons have rotations that are independent of one another. For example, orbital rotations $\times$ spin rotations (related to $\mathcal{P} = \mathcal{L} + \mathcal{S}$).

Example in book:

\[
G = \{ E, R^2 \} = C_2 \quad (\text{where } R = R_2 \text{ in}).
\]

\[
H = \{ E, I \} = S_2 \quad (\text{where } I = \text{inversion}).
\]

Note $RI = \sigma_h = \text{reflection in "horizontal" or x-y plane.}$ So,

\[
K = G \times H = C_2 \times S_2 \quad \text{contains 4 elements:}
\]

\[
\{ E, R, I, IR^2 \}
\]

\[
\cong \{ E, R, \sigma_h, \sigma_h R \} \quad \text{denoted } C_{2h}.
\]

This is isomorphic to the Viereguppe or 4-group.

Note, $\sigma_h$ acts only on z-axis (not x-y plane) \{ different spaces \}

while $R$ " " " x-y plane (not z axis)
Another example:

\[ D_3 = \{ E, R_1, \ldots, R_5 \} \]
\[ S_1 = \{ E, \sigma_h \} \]

\[ D_{3h} = D_3 \times S_1 = \{ E, \ldots, R_5, \sigma_h E, \sigma_h R_1, \ldots, \sigma_h R_5 \} \]

Class structure of product groups is simple. Let classes of \( G \) be
\[ C_i^G, C_i^H, \ldots \], each set of elements of \( G \)
and let classes of \( H \) be
\[ C_i^H, C_j^H, \ldots \] each set of elements of \( H \).

Then classes of \( G \times H \) are
\[ C_i^G \times C_j^H = \{ (g, h) : g \in C_i^G, h \in C_j^H \} \]

For example, given that classes of \( D_3 \) are \( \{ E, R, R^2, R_5, R_4, R_3 \} \)
and those of \( S_1 = \{ E, \sigma_h \} \) are \( \{ E, \sigma_h \} \), classes of \( D_{3h} \) are:
\[ \{ E \}, \{ R, R^2 \}, \{ R_3, R_4, R_5 \} \]
\[ \{ \sigma_h \}, \{ \sigma_h R_1, \sigma_h R_2 \} \]

and those of \( S_1 = \{ E, \sigma_h \} \) are \( \{ E, \sigma_h \} \), the classes of \( D_{3h} \) are:
\[ \{ E \}, \{ R, R^2 \}, \{ R_3, R_4, R_5 \} \]
\[ \{ \sigma_h \}, \{ \sigma_h R_1, \sigma_h R_2 \} \]
Now on to Ch. 3 of book, which concerns linear algebra. Much of this should be review, but it will be useful to gather it in one place.

We will deal with several kinds of rather diverse vector spaces.

Examples:

1. Ordinary 3D space, dim = 3

2. Spaces of wavefunctions, e.g. \( \{ \psi(x) \} \). These are called Hilbert spaces, dim = \( \infty \), although you may be interested in finite-dimensional subspaces.

3. Spaces of operators, for example, all operators of the form

\[
a_x J_x + a_y J_y + a_z J_z, = \vec{a} \cdot \vec{J}
\]

where \( J_x, J_y, J_z \) are the usual angular momentum operators in quan. mech. and \( \vec{a} = (a_x, a_y, a_z) \) is a real vector. We restrict \( \vec{a} \) to be real so that \( \vec{a} \cdot \vec{J} \) will be Hermitian.

4. Spaces of \( E, B \) fields.

To treat all these vector spaces at once, we adopt an abstract formalism.

Let \( V \) be a vector space,

Let \( x, y, \text{ etc} \in V \) be vectors

Let \( a, b, \text{ etc} \) be numbers (real or complex).
As you know, vectors can be added and multiplied by scalars (numbers).

Assume you are familiar with linear (in-)dependence, basis, dimensionality, subspaces, etc.

**Distinctions among vector spaces:**

1. They may be either *real* or *complex* (the only kind we will consider). For real spaces we only consider real numbers a, b, etc. when making linear combinations of vectors; for complex vector spaces we allow these coefficients to be complex. As for the vectors themselves, we don't try to say whether they are real or complex (it may not be meaningful, and in any case doesn't matter for this definition).

2. Vectors spaces may or may not have a *metric*. A metric is a complex-valued function of 2 vectors,

\[(x, y) = \text{complex \#}\]

such that:

(a) *it is linear in 2nd vector,*

\[(x, ay_1 + by_2) = a(x, y_1) + b(x, y_2)\]

and *anti-linear in 1st vector,*

\[(ax_1 + bx_2, y) = a^* (x_1, y) + b^* (x_2, y)\]
(b) $\mathbf{x}$ satisfies

$$ (x, y) = (y, x)^* $$

(c) $\mathbf{x}$ is pos. def,

$$ (x, x) = \text{real}, \geq 0 \text{ for all } x \in \mathbf{V} $$

and $$ (x, x) = 0 \iff x = 0. $$

This is the defn of a metric for a complex vector space. For a real vector space, just drop the $^*$'s in the defn above.

Example 1 above has a metric:

$$ (x, y) = x_1 y_1 + x_2 y_2 + x_3 y_3 \quad \text{(usually write } x \cdot y). $$

This is a real vector space.

Example 2:

$$ (\psi, \phi) = \int dx \, \psi^*(x) \phi(x) \quad \text{(often write } \langle \psi | \phi \rangle) $$

This is a complex vector space.

Examples 3 and 4 don't have any obvious metric, but in some circumstances it is possible to define one that is useful in physical applications. For example,

$$ (\mathbf{E}, \mathbf{E}) = \int d^3x \, |\mathbf{E}(x)|^2 = \text{const. x energy in field}. $$