Spring 2003
Physics 1190

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"Symmetry in Physics"

Web site: http://bohr.physics.berkeley.edu/1190/1192/htm

Text: Symmetry in Physics by J.P. Elliott and P.G. Dauxerre
(out of print, but chapters will be posted on web)

Course will be like 1/2 of regular lecture course (since only 2 units.)

Grade will be based on homework. There will be no exam.

Prerequisites 1. 137AB or 137B concurrently (with permission)
2. Honors standing (3.3 in UD Physics)
Introduction to Symmetry in Physics.

**Topic 1. Symmetries and Invariants.**

There is a general relation between symmetries and invariants in physics. It holds both in classical meeh and in quantum mechanics.

Classical mechanics, first.

Suppose \( V(x,y,z) = V(r) \) (a spherically or rotationally invariant potential).

Then
\[
m \dddot{x} = \dddot{y} = -\nabla V
\]

But
\[
\frac{\partial V(r)}{\partial x} = \frac{dV}{dr} \frac{\partial r}{\partial x} = \frac{dV}{dr} \frac{x}{r}.
\]

So,
\[
\begin{align*}
m \dddot{x} &= -\frac{x}{r} \frac{dV}{dr} \\
m \dddot{y} &= -\frac{y}{r} \frac{dV}{dr} \\
m \dddot{z} &= -\frac{z}{r} \frac{dV}{dr}
\end{align*}
\]

Newton's laws.

Multiply x eqn. by \(-y \) and y eqn by \(+x \) and add,

\[
m \left( x \dddot{y} - y \dddot{x} \right) = \frac{xy}{r} \frac{dV}{dr} - \frac{yx}{r} \frac{dV}{dr} = 0.
\]

But
\[
\frac{d}{dt} \left( x \dot{y} - y \dot{x} \right) = \frac{d}{dt} \left( x \dot{y} - y \dot{x} \right)
\]

So
\[
\frac{d}{dt} \left[ m \left( x \dot{y} - y \dot{x} \right) \right] = \frac{dL_2}{dt} = 0 \quad \text{where} \quad \dot{L} = \stackrel{\leftarrow}{F} \times \dot{\vec{p}}, \quad \dot{\vec{p}} = m \dot{\vec{v}}.
\]
Thus, \[ L_z = \text{const} \quad \text{if} \quad V = V(r). \]

Similarly, \( L_x, L_y \) are const too, \( \vec{L} \) (whole vector) = const.

Example of how symmetry \( \Rightarrow \) invariant. (or conservation law)

Nowadays it is believed that all the important conservation laws (such as conservation of charge) are due to symmetries of nature.

The above relationship between symmetries and invariants (in classical mechanics) can be obtained from a deeper standpoint by using Lagrangians (rather than Newton's laws). Then it is called Noether's Theorem.

Symmetries imply invariants in quantum mechanics (QM), too.

Part of the story is the following theorem (which you should have seen in 137):

Let \( H = \text{Hamiltonian} \) and suppose \([A, H] = 0\).

\[ A = \text{some operator} \]

Then \[ \frac{d}{dt} \langle A \rangle = \frac{d}{dt} \langle \psi | A | \psi \rangle = 0 \quad \text{for all} \quad \psi \rangle. \]

(assume you are familiar with Dirac bra-ket notation: \( | \psi \rangle \) is shorthand for \( \psi(x) \).)

**Proof:** \[ \frac{d}{dt} \langle \psi | A | \psi \rangle = \langle \frac{\partial}{\partial t} | \psi \rangle | A | \psi \rangle + \langle \psi | \frac{\partial}{\partial t} | \psi \rangle. \]

But \[ i \hbar \frac{\partial}{\partial t} | \psi \rangle = H | \psi \rangle \]

\[ \longrightarrow | \frac{\partial}{\partial t} | \psi \rangle \quad (\text{same thing}) \]
\[
\frac{d}{dt} \langle \psi | A | \psi \rangle = \frac{i}{\hbar} \left( - \langle \psi | H A | \psi \rangle + \langle \psi | H | A \psi \rangle \right) \\
= \frac{i}{\hbar} \langle \psi | [H, A] | \psi \rangle = 0. \quad \text{Q.E.D.}
\]

For example, \( \frac{i}{\hbar} H = \frac{p^2}{2m} + V(x) \), then \( [\mathbf{L}, H] = 0 \),

which implies \( \langle \mathbf{L} \rangle = \text{const. in time}. \) Just like in classical mechanics.

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**Topic 2**  
Symmetries, Degeneracies and Labelling of Energy Eigenstates.

Look at example of parity.  
*Def.:*  
\( (P\psi)(x) = \psi(-x) \) \quad \text{1D}  
\( (P\psi)(\vec{r}) = \psi(-\vec{r}) \) \quad \text{3D.}  

\( P = \) parity operator.  
\( P^2 = I \) or \( I \) (identity operator).  
\( P = P^* \) (Hermitian).

Now suppose \( [P, H] = 0 \).

\( \frac{i}{\hbar} H = \frac{p^2}{2m} + V(x) \) \quad \text{(1D)}  
\( \text{This means} \ V(x) = V(-x) \) \quad \text{1D}  
\( \text{or} \ V(\vec{r}) = V(\vec{-r}) \) \quad \text{3D}  

and suppose \( H\psi = E\psi \) \quad (\psi \text{ is eigenstate with energy } E).
then let \( \phi = P \psi \) or \( \phi(x) = \psi(-x) \).

Then \( H\phi = H P \psi = P H \psi = E P \psi = E \phi \).

So \( \phi \) is also (in addition to \( \psi \)) an eigenfunction of \( H \) with eigenvalue \( E \). (We have 2 functions with the same eigenvalue, \( \psi \) and \( \phi \).)

Case (a) \( \psi \) and \( \phi \) are proportional, say

\[
\phi = P \psi = c \psi \quad \text{some constant} \ c
\]

Case (b) \( \psi \) and \( \phi \) are not proportional (they are linearly independent).

In case (a), \( \psi \) is an eigenfunction of \( P \) with eigenvalue \( c \).

Thus, \( \psi \) is an eigenfunction of both \( P \) and \( H \). The constant \( c \) must be \( \pm 1 \) since \( P^2 = 1 \):

\[
P \psi = c \psi
\]

\[
p^2 \psi = c P \psi = c^2 \psi = \psi \quad \Rightarrow c^2 = 1 \quad \Rightarrow c = \pm 1.
\]

(Eigenvalues of parity operator are \( \pm 1 \)).

In case (a), the energy eigenfunction must also be an eigenstate of parity, either

\[
\psi(x) = \psi(-x) \quad (c = 1)
\]

or

\[
\psi(x) = -\psi(-x) \quad (c = -1)
\]

(even, odd state under parity).

Notice that if \( \psi \) is a nondegenerate energy eigenstate, then we must have case (a) \( (\psi \) and \( P \psi \) must be proportional, since they are both energy eigenstates with the same energy).

In case (a) we can label energy eigenstates by their parity.

(Symmetry is used to label energy eigenstates.)
In case (6), \( \phi = P \psi \) and \( \psi \) have same energy, but are linearly indep. Therefore we have a degeneracy in which degenerate states are mapped into one another by the parity operator.

More generally, it is believed that degeneracies are almost always associated with symmetries.

Example from nuclear and particle physics:

\[\begin{align*}
\text{proton} & \quad m = 938.3 \text{ MeV} \\
\text{neutron} & \quad m = 939.6 \text{ MeV}
\end{align*}\]

Both are spin \( \frac{1}{2} \) particles, both have similar interactions in nuclei (strong forces pp, pn, nn all seem to be the same) and both have nearly the same mass. What does this mean? Another example:

\[\begin{align*}
\pi^+ & \quad m = 139.6 \text{ MeV} \\
\pi^0 & \quad m = 135.0 \text{ MeV}
\end{align*}\]

A triplet of particles, all spin 0, similar in many respects.

What does this mean?

In relativity, \( E = mc^2 \), so masses of particle must be energy eigenvalues. So nearly identical masses means a near degeneracy of the Hamiltonian. Means there must be a symmetry operation that maps a proton into a neutron and vice versa, or that \( \phi \) maps the 3 pions into one another. This symmetry operation must commute with \( H \) (almost). This symmetry is called isospin, mathematically it is \( SU(2) \).

Isospin involves rotations, not in ordinary 3D space, but in an internal space.
Actually, the p, n masses are not exactly the same, so we have a mean degeneracy, which we interpret as being due to a broken symmetry.

It's like in the Zeeman effect. In the absence of a \( \vec{B} \) field, the energy eigenstates of an atom can be labelled by \( j \) and \( m_j \), the quantum numbers of \( J^2 \) and \( J_z \). These states are \( 2j+1 \)-fold degenerate, since energy doesn't depend on \( m_j \), and

\[ m_j = -j, \ldots, +j. \quad (2j+1 \text{ state}). \]

This degeneracy is due to rotational invariance. But when we turn on a \( \vec{B} \) field \( \vec{B} = B_0 \hat{z} \), the rotational invariance is broken, and the degeneracy lifts:

\[
\begin{array}{c}
(2j+1) \text{ degeneracy} \\
\overrightarrow{B} = 0 \\
\overrightarrow{B} \neq 0
\end{array}
\]

You get a triplet of nearby states (for weak \( B \)).

Similarly, for proton and neutron, we imagine

\[ H = H_0 + H_1 \]

where \( H_0 \) commutes exactly with the isospin symmetry, but \( H_1 \) does not.

\( H_0 \) is interpreted as the strong Hamiltonian

\( H_1 \) as the electromagnetic Hamiltonian.

Isospin is a symmetry of the strong interaction, not the electromagnetic. Clue to this fact is that the mass difference between \( n \) and \( p \) (or between \( \pi^+ \), \( \pi^- \)) is related to the electric charge.
Consider case of hydrogen.

\[ E_0 < E_1 < E_2 < E_3 < \ldots \]

---

Not all transitions are allowed. Selection rules tell you which ones are.

In QM it is shown that

\[
\text{transition prob.} = \text{const.} \times \left| \langle f | H_{\text{int}} | i \rangle \right|^2
\]

\[ \text{H}_{\text{int}} = \text{const.} \times \hat{E} \cdot \vec{x} \]

For electric dipole transitions, \( \text{H}_{\text{int}} = \text{const.} \times \hat{E} \cdot \vec{x} \)

Now, \( P \vec{x} P = -\vec{x} \) (point \( \vec{x} \) is odd under parity).

and \([P, H_0] = 0\) so \( |i\rangle, |f\rangle\) are states of definite parity. (+ or -)

\[ \Rightarrow \langle f | \vec{x} | i \rangle = 0 \text{ unless } |f\rangle, |i\rangle \text{ have opposite parity.} \]

\[ \Rightarrow \Delta l = \text{odd} \text{ hence } 2s \rightarrow 1s \text{ forbidden (for example).} \]
The rule \( \Delta l = \text{odd} \) is called Laporte's Rule, it was first explained (in terms of parity) by Wigner.

Now an introduction to group theory by the example of the Platonic solids (or regular polyhedra). There are 5 of these:

<table>
<thead>
<tr>
<th></th>
<th>Faces</th>
<th>Edges</th>
<th>Vertices</th>
<th>Faces made of</th>
</tr>
</thead>
<tbody>
<tr>
<td>Tetrahedron</td>
<td>4</td>
<td>6</td>
<td>4</td>
<td>triangles</td>
</tr>
<tr>
<td>Cube</td>
<td>6</td>
<td>12</td>
<td>8</td>
<td>squares</td>
</tr>
<tr>
<td>Octahedron</td>
<td>8</td>
<td>12</td>
<td>6</td>
<td>triangles</td>
</tr>
<tr>
<td>Dodecahedron</td>
<td>12</td>
<td>30</td>
<td>20</td>
<td>pentagons</td>
</tr>
<tr>
<td>Icosahedron</td>
<td>20</td>
<td>30</td>
<td>12</td>
<td>triangles</td>
</tr>
</tbody>
</table>

A rotation that maps a geometrical object unto itself is called a covering operation. It is called proper if the rotation is proper (that is, if it is a rotation you can actually realize by turning the object in your hand.)

The icosahedron has 3-fold axes through the center of each face. There are 10 of these because each 3-fold axis passes through one face and the parallel face on the opposite side. It has 6 5-fold axes through the vertices; each 5-fold axis goes through 2 opposite vertices. And it has 15 2-fold axes through the center of each edge.

An \( n \)-fold axis produces \( n-1 \) covering operations that are rotations by the non-zero angles,

\[
\frac{2\pi}{n}, \frac{4\pi}{n}, \ldots, \frac{(n-1)2\pi}{n}.
\]

In addition the identity (zero rotation) is a covering operation.
The count of covering operations (proper only) for the icosahedron is:

\[
\begin{array}{ccc}
10 & 3 \text{-fold axes} & 10 \times 2 = 20 \\
6 & 5 \text{-fold axes} & 6 \times 4 = 24 \\
15 & 2 \text{-fold axes} & 15 \times 1 = 15 \\
\text{the identity} & & 1 \\
\end{array}
\]

There are no more proper covering operations of the icosahedron. These 60 operations form the icosahedral group.

Two operations carried out in succession is considered to be the product of operations. If \( R = R_1 R_2 \), do \( R_2 \) first; then \( R_1 \), result is \( R \). The product of any 2 covering operations is another covering operation; the set of covering operations is closed under multiplication. Thus you can make a table.

Label the covering operations \( R_1, R_2, \ldots, R_{60} \).

\[
\begin{array}{cccccc}
 & \ R_1 & R_2 & R_3 & \ldots & R_{60} \\
R_1 & \ & \ & \ & \ & \\
R_2 & \ & \ & R_2 R_3 & \ & \\
R_3 & \ & \ & \ & \ & \\
\vdots & \ & \ & \ & \ & \\
R_{60} & \ & \ & \ & \ & \\
\end{array}
\]

Table gives product \( ab \), given \( a \) and \( b \).

\( 60 \times 60 = 3600 \) entries.

This is a group table.
A comment about the buckyball. Take an icosahedron, divide each edge into equal thirds with marks. Then cut off each of the 12 vertices, passing through the marks. The result is a pentagon where the vertices used to be, and hexagons where the faces used to be. The pentagons and hexagons have sides of the same length. The result is a buckyball. It is not a regular polyhedron because all the faces are not identical. But it is a closer approximation to a sphere than an icosahedron.

It has 60 vertices (the locations of the carbon atoms in C₆₀).

All vertices are "the same" (they connect 2 hexagons and one pentagon). But not all the edges are "the same", some join a pentagon and a hexagon, and some join two hexagons.

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Now Chapter 2 of book.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Here</th>
<th>The Book</th>
</tr>
</thead>
<tbody>
<tr>
<td>groups</td>
<td>G, H, K, ...</td>
<td>G, H, K, ...</td>
</tr>
<tr>
<td>elements of groups</td>
<td>g, h, a, b, ...</td>
<td>G₁, G₂, G₃, G₄, G₅, ...</td>
</tr>
<tr>
<td># of elements in a group</td>
<td>#(G)</td>
<td>g.</td>
</tr>
</tbody>
</table>
Defn. A group is a set \((G, H, \ldots)\) of elements \((g, h, a, b, \ldots)\) plus a "multiplication law" such that 4 postulates are satisfied:

1. (Closure) If \(a, b \in G\), then \(ab \in G\).
2. (Associativity) \((ab)c = a(bc)\) for all \(a, b, c \in G\).
3. (Existence of Identity) There exists a unique element \(e \in G\) (the identity) such that \(eg = ge = g\) for all \(g \in G\).
4. (Existence of Inverse) For every \(a \in G\) there exists a unique element \(a^{-1} \in G\) such that \(a^{-1}a = a^{a^{-1}} = e\).

The definition of a group doesn't say anything about whether \(ab = ba\). This is the question of commutativity (don't confuse this with associativity, all groups are associative).

If \(ab = ba\) for all \(a, b \in G\), then \(G\) is said to be Abelian.

Otherwise it is non-Abelian.

Remarks:

1. We will see later that degeneracies in quantum systems are associated with non-Abelian groups.
2. The icosahedral group is non-Abelian.
3. "Abelian" and "non-Abelian" mean the same thing as "commutative" and "non-commutative".

4. The group multiplication table for an Abelian group is symmetrical about the diagonal. For a non-Abelian group it is not symmetrical.

**Some more definitions.**

1. The order of a group $G$ is the number of elements in $G$. We will denote this by $\#(G)$.

2. A discrete group is one whose elements can be labelled with integers (counted). All groups of finite order are discrete, but there are discrete groups of infinite order, such as lattice translation in a plane (or in an infinite, 3D crystal lattice).

3. A **continuous** group is one whose elements are labelled by one or more continuous parameters. The simplest example is the group of rotations in the plane,

   \[
   \begin{array}{c}
   \text{(This group is denoted } SO(2).) \\
   \end{array}
   \]

   \[
   R(\theta) = \text{ rotation, } \quad 0 \leq \theta < 2\pi.
   \]

   $\theta = \text{ the parameter.}$

A continuous group is also called a **Lie group**.