Now we begin Hodge theory and harmonic forms. To preview the results a bit, when we add a metric to a manifold, we can do new things with differential forms and find new connections to old subjects such as cohomology groups (which do not require a metric for their definition).

If we add a metric to $M$, we can define a scalar product of wave functions,

$$\langle f_1, f_2 \rangle = \int_M \sqrt{g} \, f_1 \cdot f_2$$

where $m = \dim M$, $f_1, f_2 \in \mathcal{F}(M)$. (Real valued functions here.) Thus the wave functions make a Hilbert space. We also have interesting operators that act on these wave forms, such as the generalized Laplacian $\nabla^2$ (which requires an induced metric on $\pi^*TM$), which in lead to the notion of wave invariants of eigenfunctions.

All this (the scalar product, Laplacians, etc.) can be generalized to arbitrary $\omega$-forms. It turns out, for example, that the degeneracy of the 0 eigenvalue of $\nabla^2$ is the same as the Betti number of $M$.

The permutation or Levi-Civita symbol is familiar:

$$\epsilon_{i_1 \ldots i_m} = \begin{cases} +1 & (i_1 \ldots i_m) = \text{even permutation of } (1 \ldots m) \\ -1 & (i_1 \ldots i_m) = \text{odd permutation of } (1 \ldots m) \\ 0 & \text{otherwise} \end{cases}$$

Just because we put lower indices on it does not mean that it is a tensor.

In fact, suppose a tensor has components $\alpha_{i_1 \ldots i_m}$ in one coord. syst. $x^i$, and examine what its components are in another coord. syst. $x'^i$:
\[ T_{\mu_1 \ldots \mu_m} = \delta_{\mu_1 \ldots \mu_m} \text{ in coords } x^k \quad \text{(suppose)}.
\]

Then in coords \( y^I \),

\[ T'^{\mu_1 \ldots \mu_m} = \frac{\partial x^I}{\partial y^{\mu_1}} \ldots \frac{\partial x^I}{\partial y^{\mu_m}} E_{\nu_1 \ldots \nu_m} = \det \frac{\partial x^I}{\partial y^\nu} \delta_{\mu_1 \ldots \mu_m}. \]

So the \( \delta \)-symbol does not transform as a tensor (and we shall not call it a tensor). Now let \( g_{\mu\nu}, g'_{\mu\nu} \) be the metric in coords \( x^k \) and \( y^I \), so that

\[ g'_{\mu\nu} = \frac{\partial x^I}{\partial y^{\mu}} \frac{\partial x^J}{\partial y^{\nu}} g_{IJ}, \]

or

\[ g' = (\det \frac{\partial x^I}{\partial y^\nu}) g \quad \text{where} \quad g = \det g_{\mu\nu}, \quad g' = \det g'_{\mu\nu} \]

so

\[ \left| \det \frac{\partial x}{\partial y} \right| = \sqrt{\frac{\det g'}{\det g}}. \]

We put \( 1 \) around \( g \), since it may be negative (depends on the signature, \( \text{sgn}(g) = -1 \) in GR). Now we suppose \( M \) is orientable and we choose an orientation and work only with oriented atlases. Then \( \det \frac{\partial x}{\partial y} > 0 \), and we can drop the \( 1 \) around \( \det \frac{\partial x}{\partial y} \). Then we see that if \( T_{\mu_1 \ldots \mu_m} = \sqrt{\det g} \delta_{\mu_1 \ldots \mu_m} \)
in one coordinates, then \( T'^{\mu_1 \ldots \mu_m} = \sqrt{\det g'} E_{\nu_1 \ldots \nu_m} \) in another. We have a tensor, if we restrict to oriented coordinates. In fact it is an \( m \)-form, which we henceforth write as \( \Omega \):

\[ \Omega = \frac{1}{m!} \sqrt{\det g} E_{\mu_1 \ldots \mu_m} dx^{\mu_1} \ldots dx^{\mu_m} \]

i.e.

\[ \Omega = \sqrt{\det g} \ dx'^1 \ldots \ dx'^m \quad \text{(coord basis)} \]

or

\[ \Omega = \sqrt{\det g} \ d\theta^1 \ldots d\theta^m \quad \text{(ang basis)}. \]
Here \( \{e_i^k\} \) is any basis (coordinate or non-coordinate). Note that if \( \{e_i^k\} \) is an O.N. vielbein, then \( \sqrt{|g|} = 1 \) and \( \Omega = e_1 \cdots e_m \).

It is of interest to compute the completely contravariant components of \( \Omega \):

\[
\Omega_{\mu_1 \cdots \mu_m} = g_{\mu_1 \nu_1} \cdots g_{\mu_m \nu_m} \Omega_{\nu_1 \cdots \nu_m}
\]

\[
= \det(g^\mu_\nu) \sqrt{|g|} \varepsilon_{\mu_1 \cdots \mu_m}
\]

But \( \det g^\mu_\nu = \frac{1}{\sqrt{|g|}} = \text{sgn}(g)/|g| \). So,

\[
\Omega_{\mu_1 \cdots \mu_m} = \frac{\text{sgn}(g)}{|g|} \varepsilon_{\mu_1 \cdots \mu_m}.
\]

Useful later.

We don't worry that LHS has upper indices and RHS has lower, since \( \varepsilon \) is not a tensor.

\( \Omega \) is called the invariant volume form since its integral over any region \( R \subseteq M \) is the volume of that region in the metrical sense,

\[
\int_R \Omega = \text{vol}(R).
\]

On a space with \( m = \dim M \) dimensions, both \( r \)-forms and \( (m-r) \)-forms have the same number of independent components,

\[
\binom{m}{r} = \binom{m}{m-r}.
\]

Thus \( r \)-forms and \( (m-r) \)-forms (at a point \( x \in M \)) are vector spaces of the same dimensionality, and are isomorphic as vector spaces.
In the absence of a metric or other additional structure, however, there is no natural isomorphism between these spaces. Now, however, we will assume we have a metric \((M,g)\). Then there is a natural mapping between these spaces:

\[
\text{Hodge } \ast : \Omega^r(M) \rightarrow \Omega^{m-r}(M).
\]

It is defined by its action on the basis forms of \(\Omega^r(M)\), then extended to arb. forms by linearity. The defn is

\[
\ast (\theta^{\mu_1} \wedge \ldots \wedge \theta^{\mu_r}) = \frac{1}{(m-r)!} \Omega^{\mu_1 \ldots \mu_r} \nu_1 \ldots \nu_{m-r} (\theta^{\nu_1} \wedge \ldots \wedge \theta^{\nu_{m-r}}).
\]

Indices on \(\Omega\) are raised with \(g^{\nu_1}\).

As a special case, consider the 0-form \(1 \in \Omega^0(M)\) (constant scalar). Then \(r=0\) in the above, and we have

\[
\ast 1 = \frac{1}{m!} \Omega^{\nu_1 \ldots \nu_m} \theta^{\nu_1} \wedge \ldots \wedge \theta^{\nu_m} = \Omega,
\]

\[
\ast 1 = \Omega
\]

The defn above makes it clear that \(\ast\) is linear, but is it an isomorphism (i.e., is it invertible)? We answer by computing \(\ast \ast\), a map: \(\Omega^r(M) \rightarrow \Omega^r(M)\). We apply defn above twice, get

\[
\ast \ast (\theta^{\mu_1} \wedge \ldots \wedge \theta^{\mu_r}) = \frac{1}{(m-r)!} \Omega^{\mu_1 \ldots \mu_r} \nu_1 \ldots \nu_{m-r}
\]

\[
\times \frac{1}{r!} \Omega^{\nu_1 \ldots \nu_{m-r}} \lambda_1 \ldots \lambda_r (\theta^{\lambda_1} \wedge \ldots \wedge \theta^{\lambda_r}).
\]
Transform this. First raise and lower \( \nu_1 \ldots \nu_{m-r} \) indices to make indices uniformly upper or lower. Next, on \( \Omega \nu_1 \ldots \nu_{m-r} \lambda_1 \ldots \lambda_r \), migrate \( \lambda \) indices to left of \( \nu \) indices. This involves \((m-r)r\) sign changes, so

\[
\Omega \nu_1 \ldots \nu_{m-r} \lambda_1 \ldots \lambda_r = (-1)^{r(m-r)} \Omega \lambda_1 \ldots \lambda_r \nu_1 \ldots \nu_{m-r}.
\]

Thus,

\[
x \cdot (\theta^\mu_1 \wedge \ldots \wedge \theta^\mu_r) = \frac{1}{(m-r)!} \frac{1}{r!} (-1)^{r(m-r)} \Omega \mu_1 \ldots \mu_r \nu_1 \ldots \nu_{m-r} \Omega \lambda_1 \ldots \lambda_r \nu_1 \ldots \nu_{m-r} \theta^{\lambda_1} \wedge \ldots \wedge \theta^{\lambda_r}
\]

\[
\Rightarrow = \frac{\text{sgn}(g)}{V(g)} \epsilon_{\mu_1 \mu_2 \ldots \mu_r} \nu_1 \nu_2 \ldots \nu_{m-r} \times V(g) \epsilon_{\lambda_1 \lambda_2 \ldots \lambda_r} \nu_1 \nu_2 \ldots \nu_{m-r}
\]

\[
= \text{sgn}(g) \text{sgn}(\mu_1 \ldots \mu_r) (m-r)!
\]

where we use identities for products of two \( \epsilon \)'s and where

\[
\text{sgn}(\mu_1 \ldots \mu_r) = \begin{cases} 
\pm 1 & \text{if } (\lambda_1 \ldots \lambda_r) \text{ is (even) product of } \mu_1 \ldots \mu_r \\
0 & \text{otherwise}
\end{cases}
\]

Thus,

\[
\text{sgn}(\mu_1 \ldots \mu_r) \theta^{\lambda_1} \wedge \ldots \wedge \theta^{\lambda_r}
\]

\[
= r! \theta^{\mu_1} \wedge \ldots \wedge \theta^{\mu_r}.
\]
Putting it all together, we have

\[
** \left( \theta^{\mu_1} \wedge \ldots \wedge \theta^{\mu_r} \right) = \text{sgn}(g) (-1)^{r(m-r)} \left( \theta^{\mu_1} \wedge \ldots \wedge \theta^{\mu_r} \right).
\]

or

\[
** = \text{sgn}(g) (-1)^{r(m-r)}
\]

when acting on \( \omega \in \Omega^r(M) \).

Equivalently,

\[
x^{-1} = \text{sgn}(g) (-1)^{r(m-r)} * \]

Thus \(*\) is invertible, and \(*\) is an isomorphism.

Now consider the interaction of \(*\) with \( \wedge \). Let \( \alpha, \beta \in \Omega^r(M) \).

Then \(*\beta\) is an \((m-r)\)-form, and

\[
\alpha \wedge *\beta \in \Omega^m(M).
\]

Thus \(\alpha \wedge *\beta\) must be proportional to the volume form \(\Omega\), i.e., it must be \(f \Omega\) for some scalar \(f\). Now we work out what \(f\) is.

Write

\[
\alpha = \frac{1}{r!} \alpha_{\mu_1 \ldots \mu_r} \theta^{\mu_1} \wedge \ldots \wedge \theta^{\mu_r}
\]

\[
\beta = \frac{1}{r!} \beta_{\nu_1 \ldots \nu_r} \theta^{\nu_1} \wedge \ldots \wedge \theta^{\nu_r}
\]

Hence

\[
*\beta = \frac{1}{r!} \beta_{\nu_1 \ldots \nu_r} \left( \frac{1}{(m-r)!} \Omega_{\nu_1 \ldots \nu_r} \lambda_{1 \ldots \lambda_{m-r}} \theta^{\lambda_1} \wedge \ldots \wedge \theta^{\lambda_{m-r}} \right),
\]

and

\[
\alpha \wedge *\beta = \frac{1}{(r-1)! (m-r)!} \alpha_{\mu_1 \ldots \mu_r} \beta_{\nu_1 \ldots \nu_r} \Omega_{\nu_1 \ldots \nu_r} \lambda_{1 \ldots \lambda_{m-r}} \theta^{\lambda_1} \wedge \ldots \wedge \theta^{\lambda_{m-r}}
\]

\[\times \theta^{\mu_1} \wedge \ldots \wedge \theta^{\mu_r} \wedge \theta^{\nu_1} \wedge \ldots \wedge \theta^{\nu_r} \wedge \theta^{\lambda_1} \wedge \ldots \wedge \theta^{\lambda_{m-r}}.\]
Transform this. First raise and lower \( \gamma \), \( \nu \) indices, use
\[
\theta^{\mu_1 \ldots \nu_1 \ldots \theta^{\nu_r} \ldots \theta^{\alpha_m} = \epsilon_{\mu_1 \ldots \mu_r \ldots \alpha_m \ldots \alpha_n} \theta^1 \ldots \theta^m.
\]
Get,
\[
\alpha^{\mu_1 \ldots \nu_1 \ldots \nu_r} \epsilon_{\mu_1 \ldots \mu_r \ldots \nu_1 \ldots \nu_r} \sqrt{|g|} \epsilon_{\nu_1 \ldots \nu_r \ldots \alpha_m \ldots \alpha_n} \\
\times \epsilon_{\mu_1 \ldots \mu_r \ldots \alpha_m \ldots \alpha_n} \theta^1 \ldots \theta^m
\]
\[
= \frac{1}{(r!)_2^2 (m-r)!} \alpha^{\mu_1 \ldots \mu_r} \epsilon_{\nu_1 \ldots \nu_r} (m-r)! \text{sgn} (\mu_1 \ldots \mu_r) \Omega
\]
\[
\alpha^{\mu_1 \ldots \mu_r} \beta^{\nu_1 \ldots \nu_r} = (\frac{1}{r!} \alpha^{\mu_1 \ldots \mu_r} \beta^{\nu_1 \ldots \nu_r}) \Omega
\]
The scalar multiplying \( \Omega \) is the complete contraction of the components of \( \alpha \) with those of \( \beta \).

Several things to note about this. First, the answer is symmetric in \( \alpha, \beta \), so
\[
\alpha \wedge \beta = \beta \wedge \alpha
\]
Next, if \( g \) is pos. def., then \( \epsilon_{\mu_1 \ldots \mu_r} \alpha^{\mu_1 \ldots \mu_r} \geq 0 \), i.e., you get a pos. def. scalar product of \( \omega \)-forms. Define
\[
\langle \alpha, \beta \rangle = \int_M \alpha \wedge \beta.
\]
Then if \( g \) is pos. def. (a Riemannian manifold) then this scalar product is also pos. def., i.e., \( \langle \alpha, \alpha \rangle \geq 0 \) and \( \langle \alpha, \alpha \rangle = 0 \) iff \( \alpha = 0 \).
Notice that if \( \alpha, \beta \) are 0-forms (call them \( f_1, f_2 \)), then we get the obvious scalar product of them,

\[
\langle f_1, f_2 \rangle = \int \Omega f_1 f_2 = \int d^m x \sqrt{|g|} f_1 f_2.
\]

More generally, if \( g \) is pos. def., we have a scalar product on \( \Omega^r(M) \) that allows us to define a Hilbert space of \( r \)-forms. The functional analysis of this is easiest in the case of compact \( M \).

An example of this scalar product. Let \( F \) be the EM field tensor in 4D space time, (maybe curved),

\[
F = \frac{i}{2} F_{\mu\nu} \theta^\mu \wedge \theta^\nu.
\]

Then

\[
\langle F, F \rangle = \int F \wedge \ast F = \frac{1}{2} \int F_{\mu\nu} F^{\mu\nu} \sqrt{|g|} d^4 x.
\]

This is \( 2 \times \) the EM field action,

\[
S_{\text{EM}} = \frac{1}{2} \langle F, F \rangle.
\]

(But the scalar product is not pos. def. on space-time.)

Now consider interaction of \( \ast \) with exterior deriv. \( d \).

Let \( \alpha \in \Omega^r(M), \beta \in \Omega^{r-1}(M) \). Then \( \langle \alpha, d\beta \rangle \) is meaningful.

We define the operator \( d^+ \) (the adjoint of \( d \)) by

\[
\langle \alpha, d\beta \rangle = \langle d^+\alpha, \beta \rangle, \quad \forall \alpha \in \Omega^r(M), \ \beta \in \Omega^{r-1}(M).
\]

\( d^+ \) is the unique operator that makes this equation true. Note that

\[
d^+: \Omega^r(M) \rightarrow \Omega^{r-1}(M)
\]

\[
d: \Omega^{r-1}(M) \rightarrow \Omega^r(M)
\]
(d and $d^+$ work in opposite directions).

We can find an expression for $d^+$ as follows.

$$
\langle d^+ \alpha, \beta \rangle = \langle \alpha, d^\beta \rangle = \langle d\beta, \alpha \rangle = \int_M d\beta^{\wedge} \alpha.
$$

But $d(\beta^{\wedge} \alpha) = d\beta^{\wedge} \alpha + (-1)^{n-1} \beta^{\wedge} d^\alpha$, so

$$
\Rightarrow \int_M d(\beta^{\wedge} \alpha) = (-1)^{n-1} \int_M \beta^{\wedge} d^\alpha.
$$

First term vanishes by Stokes' Theorem (we assume $\partial M = 0$), so

$$
\Rightarrow = (-1)^r \int_M \beta^{\wedge} d^\alpha = (-1)^r \int_M \beta^{\wedge} \ast (-1)^{m} d^\alpha
$$

$$
= (-1)^r \langle \beta, (-1)^{m} d^\alpha \rangle = (-1)^r \langle \ast^{-1} d^\alpha, \beta \rangle.
$$

This implies,

$$
\boxed{d^+ = (-1)^r \ast^{-1} d^\ast} \quad \text{acting on r-forms.}
$$

In this expression, $\ast^{-1}$ acts on an $(m-r+1)$-form, so

$$
\ast^{-1} = \text{sgn}(g) (-1)^{(m-r+1)(r-1)} 
\ast.
$$

Since

$$
\Rightarrow r + (m-r+1)(r-1) \equiv mr + m + 1 \pmod{2},
$$

we have

$$
\boxed{d^+ = \text{sgn}(g) (-1)^{mr + m + 1} \ast d^\ast} \quad \text{acting on r-forms.}
$$
We note the identity,
\[ d^* d^+ = 0 \]

which is easily proved, \( \sqrt{\text{sign of } \ast \ast} \).

\[ d^* d^+ = \ast \ast \ast d^\ast = \pm \ast d d^\ast = 0. \]

Note that \( d^+ \) annihilates any 0-form,
\[ d^f = 0, \quad f \in \mathcal{F}(\mathcal{M}) \]

because there are no \((-1)\)-forms.
Summary:

$$\Omega = \sqrt{\text{sgn}(g)} \epsilon^{\mu_1 \ldots \mu_m} = \frac{\sqrt{\text{sgn}(g)}}{m!} \epsilon_{\mu_1 \ldots \mu_m} \theta^{\mu_1 \ldots \mu_m}$$

$$\Omega_{\mu_1 \ldots \mu_m} = \sqrt{\text{sgn}(g)} \epsilon_{\mu_1 \ldots \mu_m}$$

$$\Omega^{\mu_1 \ldots \mu_m} = \frac{\text{sgn}(g)}{\sqrt{\text{sgn}(g)}} \epsilon_{\mu_1 \ldots \mu_m}$$

$$\Omega = \star 1$$

$$\star : \Omega^r(M) \to \Omega^{m-r}(M)$$

$$\star (\theta^{\mu_1 \ldots \mu_r}) = \frac{1}{(m-r)!} \Omega_{\nu_1 \ldots \nu_{m-r}} \epsilon^{\nu_1 \ldots \nu_{m-r}} (\theta_{\nu_1 \ldots \nu_{m-r}})$$

$$\star^2 = \text{sgn}(g) (-1)^{r(m-r)} \star$$ \quad on $\omega \in \Omega^r(M)$

$$\star^{-1} = \text{sgn}(g) (-1)^{r(m-r)} \star$$

$$\alpha \wedge \star \beta = \left( \frac{1}{r!} \epsilon^{\mu_1 \ldots \mu_r} \beta_{\mu_1 \ldots \mu_r} \right) \Omega, \quad \alpha, \beta \in \Omega^r(M)$$

$$\alpha \wedge \star \beta = \beta \wedge \star \alpha$$

$$\langle \alpha, \beta \rangle = \int_M \alpha \wedge \star \beta = \langle \beta, \alpha \rangle \quad \text{(pos def. if } g \text{ pos def.)}$$

$$\langle \alpha, d\beta \rangle = \langle d^\dagger \alpha, \beta \rangle, \quad \forall \alpha \in \Omega^r(M), \beta \in \Omega^{r-1}(M).$$

$$d^+: \Omega^r(M) \to \Omega^{r-1}(M)$$

$$d: \Omega^r(M) \to \Omega^{r+1}(M)$$

$$d^+ = (-1)^{r-1} \star d \star = \text{sgn}(g) (-1)^{mr+r+1} \star d \star. \quad \text{(on } \omega \in \Omega^r(M))$$

$$d^+ d^+ = 0$$
Now work out the action of $d^+$ on a 1-form $\alpha \in \Omega'(M)$. (We know $d^+$ annihilates 0-forms). $d^+\alpha$ is a scalar, want to find it. Write $\alpha = \alpha_\mu \theta^\mu$.

First compute $*\alpha$:

$$*\alpha = \frac{\alpha_\mu}{(m-1)!} \Omega^{\mu \nu_1 \ldots \nu_m} (\theta^{\nu_1} \wedge \ldots \wedge \theta^{\nu_m}) \quad \text{(raise + lower } \mu)$$

$$= \frac{1}{(m-1)!} \alpha^\mu \Omega_{\mu \nu_1 \ldots \nu_m} (\theta^{\nu_1} \wedge \ldots \wedge \theta^{\nu_m})$$

$$\rightarrow \sqrt{\lvert g \rvert} \, \varepsilon_{\nu_1 \ldots \nu_m}.$$

$$d*\alpha = \frac{1}{(m-1)!} \left( \sqrt{\lvert g \rvert} \, \alpha^\mu \right)_\sigma \varepsilon_{\mu \nu_1 \ldots \nu_m} \Theta^\sigma \wedge \theta^{\nu_1} \wedge \ldots \wedge \theta^{\nu_m}$$

$$\rightarrow \varepsilon_{\sigma \nu_1 \ldots \nu_m} \Theta^\sigma \wedge \theta^{\nu_1} \wedge \ldots \wedge \theta^{\nu_m}$$

$$\rightarrow \frac{\omega}{\sqrt{\lvert g \rvert}}.$$

$$d*\alpha = \frac{1}{(m-1)!} \left( \sqrt{\lvert g \rvert} \, \alpha^\mu \right)_\sigma \left( \sqrt{\lvert g \rvert} \, \alpha^\mu \right)_\mu \Omega.$$

Note: $\delta^\sigma_\mu = \delta_\mu^\sigma$. More generally,

$$\delta^\sigma_\mu = \begin{vmatrix} \delta^{\sigma_1}_\mu & \ldots & \delta^{\sigma_t}_\mu \\ \delta^{\sigma_1}_{\mu_1} & \ldots & \delta^{\sigma_t}_{\mu_1} \\ \vdots & \ddots & \vdots \\ \delta^{\sigma_1}_{\mu_t} & \ldots & \delta^{\sigma_t}_{\mu_t} \\ \delta^{\sigma_1}_{\mu_t} & \ldots & \delta^{\sigma_t}_{\mu_t} \end{vmatrix}$$

$$\rightarrow \frac{1}{\sqrt{\lvert g \rvert}} \left( \sqrt{\lvert g \rvert} \, \alpha^\mu \right)_\mu \Omega.$$
A digression on a useful theorem. Let $X = X^\mu e_\mu$ be a vector field. Then

$$X^\mu, \mu = \frac{1}{\sqrt{\det g}} \left( \sqrt{\det g} X^\mu \right)_{,\mu}$$

is a useful formula.

We assume Levi-Civita connections.

We used this theorem in computing field terms from Lagrangian.

To prove it, expand RHS,

$$\text{RHS} = \frac{1}{\sqrt{\det g}} \left( \sqrt{\det g} X^\mu + X^\mu_{,\mu} \right)$$

But by the formula for the derivative of a determinant,

$$= \frac{1}{2} \left( \frac{\partial g}{\partial \alpha^\alpha} g_{\alpha^\beta, \mu} \right) X^\mu + X^\mu_{,\mu}.$$

Now

$$X^\mu_{,\nu} = X^\mu_{,\nu} + \Gamma_{\alpha \nu}^\mu X^\alpha,$$

so

$$\text{LHS} = X^\mu_{,\mu} = X^\mu_{,\mu} + \Gamma_{\alpha \mu}^\mu X^\alpha.$$

Also,

$$\Gamma_{\alpha \beta}^\mu = \frac{1}{2} g^{\mu \nu} \left( g_{\nu \alpha, \beta} + g_{\nu \beta, \alpha} - g_{\alpha \beta, \nu} \right),$$

so

$$\Gamma_{\alpha \mu}^\mu = \frac{1}{2} g^{\mu \nu} \left( g_{\nu \alpha, \mu} + g_{\nu \mu, \alpha} - g_{\alpha \mu, \nu} \right),$$

(two terms cancel by exchange $\mu \leftrightarrow \nu$ and symmetry)

$$\Gamma_{\alpha \mu}^\mu = \frac{1}{2} g^{\mu \nu} g_{\mu \nu, \alpha}.$$

So

$$\text{LHS} = X^\mu_{,\mu} + \frac{1}{2} g^{\mu \nu} g_{\mu \nu, \alpha} X^\alpha = \text{RHS}.$$  

$\text{QED}$
So to go back, we have

\[
d \star \alpha = \frac{1}{\sqrt{|g|}} \left( \sqrt{|g|} \alpha^\mu \right) \Omega = \alpha^\mu \; \mu \Omega
\]

Now apply \((-1)^{n-1} = -\star^{-1}\), note that \(\Omega = \star 1\) so \(\star^{-1} \Omega = 1\), get

\[
d^\ast \alpha = -\star^{-1} d \star \alpha = -\alpha^\mu \; \mu
\]

\[
d^\ast \alpha = -\alpha^\mu \; \mu
\]

It is the covariant "divergence" of \(\alpha\) (converted to a vector via \(g\)).

**Note:** In special case \(\alpha = df\) \((f \in \Omega^0(M))\),

we have

\[
d^\ast df = -f^\mu \; \mu
\]

This is (minus) the obvious generalization of the Laplacian to curved spaces,

\[
-\nabla^2 f = -f_{\mu \nu} \quad \text{on Euclidean } \mathbb{R}^n.
\]

We'll write another note on this result:

\[
\int_M d^M x \sqrt{|g|} \alpha^\mu \; \mu = -\int_M (d^\ast \alpha) \Omega = -\int_M d^\ast \alpha \wedge \star 1
\]

\[
= -\langle d^\ast \alpha, 1 \rangle = -\langle \alpha, d1 \rangle = 0.
\]

A more straightforward way to see the same thing is to use integration by parts,
\[ \int d^m x \sqrt{\text{det} g} \, \alpha^\mu; \mu = \int d^m x \left( \sqrt{\text{det} g} \, \alpha^\mu \right), \mu = 0. \]

You have to convert a covariant deriv. to an ordinary deriv. if you want to integrate by parts.

Now Hodge * and Maxwell eqns (E+M). Already noted,

\[ S_{EM} = \langle F, F \rangle = \int F \wedge * F. \]

Maxwell eqns in SR:

\[ F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad \text{or} \quad F = dA, \quad F = \frac{1}{2} F_{\mu\nu} dx^\mu \wedge dx^\nu \]

\[ F_{[\mu\nu;\sigma]} = 0 \quad \text{or} \quad dF = 0 \]

\[ F_{\mu\nu,\nu} = J^\mu \]

\[ J^\mu;\nu = 0 \]

We use the comma goes to semicolon rule to put these into GR. For example,

\[ F_{[\mu\nu;\sigma]} = 0. \]

Question: does this still mean \( dF = 0 \)? (There are extra terms involving \( \Gamma \) from the covariant derivatives). Answer: Yes, because in the LC connection, all the \( \Gamma \) terms cancel when computing the components of an exterior derivative of any tensor-valued (real-valued, i.e., not Lie algebra-valued) form. Thus,

\[ F_{[\mu\nu;\sigma]} = 0 \Rightarrow F_{[\mu\nu,\sigma]} = 0 \Rightarrow dF = 0 \Rightarrow \]

\[ \Rightarrow F = dA \quad (\text{Poincaré lemma}) \Rightarrow F_{\mu\nu} = \Lambda_{\mu}^\rho A_{\rho,\mu} - A_{\mu,\nu} \]

\[ \Rightarrow F = A_{\nu,\mu} - A_{\mu,\nu}. \]
As for $J^\mu, \mu = 0$ (in SR), it becomes $J^\mu, \mu = 0$ (in GR).

Now define

$$J = J_\mu dx^\mu \quad \text{(current 1-form)},$$

and charge conservation becomes

$$d^+ J = 0.$$

Finally, as for $F^{\mu\nu}, \nu = J^\mu$ (in SR), it becomes $F^{\mu\nu}, \nu = J^\mu$ (in GR). It can be shown (exercise for you) that this is equivalent to

$$d^+ F = J,$$

which is consistent with charge conservation because $d^+ J = d^+ d^+ F = 0$.

Summary of Maxwell eqns:

$$\begin{align*}
F &= dA, \quad d^+ J = 0. \\
dF &= 0 \\
d^+ F &= J
\end{align*}$$

We can use these eqns to get a wave eqn. for $A$:

$$d^+ F = d^+ dA = J.$$

Although $d^+ d$ acting on scalars (we saw above) is (minus) the covariant Laplacian (i.e., d’Alembertian in space-time), this is not quite true for forms of higher rank ($r \geq 1$). For all forms we define,

$$\Delta = d^+ d + d d^+.$$
This agrees with the case of scalars since \( d^* f = 0 \) (any scalar \( f \)), so
\[
\Delta f = d^* df.
\]

But on a 1-form such as \( A \) we have
\[
\Delta A = J - dd^* A.
\]
The term on the RHS vanishes if we choose Lorentz gauge, \( d^* A = 0 \).

[Think: \(-\nabla \cdot \vec{A} = \frac{\partial}{\partial t} - \nabla \times (\nabla \times \vec{A})\) in NR magnetostatics.]

Now we explore the properties of the operator \( \Delta \).
\[
\Delta = d^* d + dd^* \quad \text{(def)}
\]
\[
= (d + d^*)^2 \quad \text{since} \quad d^2 = d^* d^* = 0.
\]

In the following we assume the Riemannian case, so \( g \) is positive def.

This means that \( \langle \cdot, \cdot \rangle \) is also positive def., so that
\[
\langle \alpha, \alpha \rangle \geq 0, \quad \text{and} \quad \langle \alpha, \alpha \rangle = 0 \iff \alpha = 0. \quad \text{(any form \( \alpha \))}
\]

First note that \( \Delta \) is \emph{\( \mathbb{C} \)} Hermitian,
\[
\langle \alpha, \Delta \beta \rangle = \langle \alpha, d^* d \beta \rangle + \langle \alpha, dd^* \beta \rangle
\]
\[
= \langle d^* \alpha, d \beta \rangle + \langle d \alpha, d^* \beta \rangle
\]
\[
= \langle d^* d \alpha, \beta \rangle + \langle dd^* \alpha, \beta \rangle
\]
\[
= \langle \Delta \alpha, \beta \rangle.
\]

(More simply, just use the rules of \( \mathcal{T} \) on products of operators, as in \( \mathcal{G} \).

Next note that \( \Delta \) is a \textbf{positive definite} \textit{nonnegative} definite operator,

i.e., \( \langle \alpha, \Delta \alpha \rangle \geq 0 \ \forall \alpha \). Proof is easy: