Then

\[ \text{for } i \geq 0, \text{ more precisely } \]

is the i-th face, a singular 2-cube. This defines \( \partial : C_i(M) \to C_{i-1}(M) \)

Then we define, as in homology theory,

\[
Z_r(M) = \{ c \in C_r(M) \mid \partial c = 0 \} = \text{r-th cycle group} = \text{ker} \partial_r
\]

\[
B_r(M) = \{ c \in C_r(M) \mid c = \partial b, \text{ some } b \in B_{r+1} \} = \text{r-th boundary group} = \text{im} \partial_{r+1}.
\]

And we have \( \partial^2 = 0 \), as before, so \( B_r(M) \subset Z_r(M) \). And we define

\[
H_r(M) = \frac{Z_r(M)}{B_r(M)} = \text{r-th homology group}
\]

This is the same group as before.

The properties of \( \partial \) on chains is mirrored in the properties of \( d \) acting on forms. The terminology reflects this:

\[
\Omega^r(M) = \{ \text{r-forms on } M \}
\]

Closed forms \( \rightarrow Z^r(M) = \{ \omega \in \Omega^r(M) \mid d\omega = 0 \} \), r-th cocycle group.

Exact forms \( \rightarrow B^r(M) = \{ \omega \in \Omega^r(M) \mid \omega = d\beta, \text{ some } \beta \in \Omega^{r-1}(M) \} \), \text{im} \partial_{r-1} \text{ r-th coboundary group}

And because \( d^2 = 0 \), we have \( B^r(M) \subset Z^r(M) \). And we define

\[
H^r(M) = \frac{Z^r(M)}{B^r(M)} = \frac{\text{closed}}{\text{exact}} = \frac{\text{cocycles}}{\text{coboundaries}} = \text{r-th cohomology group}
\]
To explore this association, we need Stokes' theorem, which says, if \( c \subseteq C^{1+}_0(M) \), \( \omega \in \Omega^r(M) \),

\[
\int_c \omega = \int_c d\omega
\]

To prove this it suffices to consider a single singular r-cube, since chains are lin. combs. of such things. We do example of 3-forms. Let \( \dim M = m = \) anything. Let \( \omega \in \Omega^2(M) \), so \( d\omega \in \Omega^3(M) \).

Let \( \alpha = f^*\omega \), \( \alpha \in \Omega^2(\mathbb{R}^3) \). This means \( \alpha \) has 3 nonzero components,

\[
\alpha = \alpha_x \, dx \wedge dy + \alpha_y \, dy \wedge dz + \alpha_z \, dz \wedge dx
\]

\[
d\alpha = d(f^*\omega) = f^*(d\omega)
\]

\[
= \left( \frac{\partial \alpha_x}{\partial x} + \frac{\partial \alpha_y}{\partial y} + \frac{\partial \alpha_z}{\partial z} \right) dx \wedge dy \wedge dz
\]

So,

\[
\int_c d\omega = \int_{I^3} d\alpha = \int_0^1 dx \int_0^1 dy \int_0^1 dz \quad (\text{)} = 3 \text{ terms}
\]

Look at \( z \)-term,

\[
\int d\omega = \int d\alpha = \int_0^1 dx \int_0^1 dy \left[ \alpha_z(x, y, 1) - \alpha_z(x, y, 0) \right].
\]
we get 6 terms altogether for $\int_C dw$. Now consider $\int_C \omega$.

Look at the top face of the cube: let $g_i : I^2 \to \text{top face of } I^3$.

$x = u$
$y = v$
$z = l$

Then $g_i^* \omega = g_i^* f^* \omega = g_i^* \alpha$. But

$g_i^* \alpha = \int_{I^2} \alpha_2(x,y) \, dx \, dy = \alpha_2(u,v) \, du \, dv$,

since $dz = 0$ on top face. Thus,

$$\int_{I^2} (f \circ g_i)^* \omega = \int_{I^2} g_i^* \alpha = \int_0^1 \int_0^1 \alpha_2(u,v) \, du \, dv.$$

This is one of the 6 terms from the integral $\int_C dw$. The other 5 add up to make $\int_C dw$.

---

<table>
<thead>
<tr>
<th>homology</th>
<th>cohomology</th>
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<tbody>
<tr>
<td>$C_r(M)$</td>
<td>$\Omega^r(M)$</td>
</tr>
<tr>
<td>$Z_r(M)$</td>
<td>$Z^r(M)$</td>
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<tr>
<td>$B_r(M)$</td>
<td>$B^r(M)$</td>
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<tr>
<td>$H_r(M)$</td>
<td>$H^r(M)$</td>
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</tbody>
</table>
These spaces are dual to each other, in a certain sense. Notation, let \( \omega \in \Omega^r(M) \), \( c \in C_\tau(M) \), then write
\[
\int_c \omega = (\omega, c) \in \mathbb{R}.
\]

Thus \( r \)-forms are real-valued, linear operators on the space of \( r \)-chains, and vice versa. This suggests that maybe \( \Omega^r(M) \) is the dual space to \( C_\tau(M) \),
\[
\Omega^r(M) = C_\tau(M)^*.
\]

These are \( \infty \)-dimensional vector spaces, so making this interpretation precise involves a big effort. We will just proceed as if it is true.

In an earlier HW problem, had vector space \( V \), its dual \( V^* \), a subspace \( U \subset V \), and \( X^* \subset V^* \), where \( X^* \) is set of forms that annihilate \( U \). Then we had a theorem,
\[
\dim U + \dim X^* = \dim V \quad (\ker X^* = U)
\]
so if \( U \) is big (high dimensionality), \( X^* \) is small and vice versa. (You can specify a vector subspace \( U \subset V \) either by vectors that span it, or by forms that annihilate it.)

So what are the forms in \( \Omega^r(M) \) that annihilate \( Z_r(M) \subset C_\tau(M) \)? (Note if they annihilate \( Z_r \), they annihilate \( B_r \), too). Answer: \( B_\tau^r(M) \) (coboundaries, or exact forms). How to see: let \( \beta \in B_\tau^r(M) \), \( \beta \in Z_r(M) \) [\( \beta = \text{exact} \), \( \beta = \text{a cycle} \)]. Then \( (\beta = d\gamma, \text{ some } \gamma \in \Omega^{r-1}(M)) \).
\[
\int_\beta = \int_{\partial \gamma} \beta = \int_\gamma = 0.
\]
And what are the forms that annihilate \( B_r(M) \subset Z_r(M) \subset C_r(M) \)?

Ans: \( Z^r(M) \) (cocycles, or closed forms). How to see: Let \( b \in B_r(M) \) (a boundary, so \( b = \partial c \), some \( c \in C^{r+1}(M) \)), and let \( \xi \in Z^r(M) \), (a closed form, \( d\xi = 0 \)). Then

\[
\int_b \xi = \int_{\partial c} \xi = \int_c d\xi = 0.
\]

Conversely, interpreting \( C_r(M) \) as the operators and \( C^r(M) \) as the operands, then \( B_r(M) \) is the space that annihilates \( Z^r(M) \), and \( Z_r(M) \) annihilates \( B^r(M) \).

What is the space dual to \( H_r(M) \) (homology group)?

An element of \( H_r(M) \) is \([z]\) where \( z \in Z_r(M) \) is a cycle and \([z] = [z + b]\) where \( b \in B_r(M) \) is a boundary. So, an operator acting on \( H_r(M) \) would be one that acts on \( Z_r(M) \) but annihilates boundaries, so the answer does not depend on which cycle \( z \) in \([z]\) is chosen. This means it should be a cocycle, because if \( \xi \in Z^r(M) \), then

\[
(\xi, z + b) = (\xi, z) + (\xi, b).
\]

So, \( \xi \in Z^r(M) \) can be associated with an element of \( H_r(M)^* \).

(You can think of \( (\xi, [z]) \).) However, this element of \( H_r(M)^* \) is not uniquely specified by \( \xi \), because \( \xi' = \xi + \beta \), where \( \beta \in B^r(M) \), \( \beta = dy \), specifies the same map: \( H_r(M) \to \mathbb{R} \):

\[
(\xi + \beta, z) = (\xi, z) + (\beta, z) \hspace{1cm} \rightarrow \hspace{1cm} (dy, z) = 0.
\]
Thus, the element of $H_r(M)^*$ is specified by an equivalence class $[\xi] = [\xi + \beta]$, $\beta = dy$, that is, an element of $H^r(M)$. This suggests that

$$H_r(M)^* = H^r(M).$$

De Rham's theorem asserts that this is correct, and moreover that in the case $M$ is compact, $H^r(M)$ is finite-dimensional. This dimensionality is the $r$-th Betti number,

$$b_r = \dim H_r(M) = \dim H^r(M).$$

$H^r(M)$ is properly called the $r$-th de Rham cohomology group.

Remark: In Stokes' Theorem,

$$(\omega, \delta c) = (d\omega, c)$$

we can see that $d$ is the pull-back of $\delta$. That is,

$$\delta_r : C^r(M) \to C^r_1(M)$$

$$\delta_r^* : C^r_1(M)^* \to C^r(M)^*$$

$$\Omega^r(M) \to \Omega^r_1(M)$$

Thus $d_{r-1} = \delta_r^*$. Nakahara mistakenly calls this the adjoint (which requires a metric).
An example that will illustrate how topological information is contained in differential forms. Consider a magnetic monopole,

$$\overrightarrow{B} = g \overrightarrow{\hat{r}} \frac{1}{r^2}.$$ 

Magnetic fields are closely associated with 2-forms, because you integrate $\overrightarrow{B}$ over 2D surfaces to get fluxes. Let $d\hat{a}$ be an area element subtending solid angle $d\Omega$ at the origin,

then by geometry,

$$r^2 d\hat{a} = r^2 d\Omega;$$

hence $\overrightarrow{B} \cdot d\hat{a} = g d\Omega$. To put this in the language of diff. forms, write $\beta$ instead of $\overrightarrow{B} \cdot d\hat{a}$, and write $d\Omega$ in spherical coordinates:

$$\beta = g \sin \theta d\theta d\phi.$$ 

Since $\theta$ and $\phi$ are not continuous everywhere, it's not obvious that this is a smooth 2-form. So transform to Cartesian coordinates. You find

$$\beta = g \frac{x dy dz + y dz dx + z dx dy}{r^3},$$

which is obviously smooth everywhere except $r=0$, the location of the monopole. So to work with smooth fields, define

$$M = \mathbb{R}^3 - \{0\}.$$ 

Then $\beta \in \Omega^2(M)$. In fact, $d\beta = 0$, so $\beta \in \Omega^2(M)$. 
(You can compute $d\beta$ directly.) Since $d\beta = 0$, $[\beta]$ is an element of $H^2(M)$. It is a non-trivial element (non-zero) of $H^2(M)$, because $\beta$ is not exact. To see this, consider the integral of $\beta$ over the sphere $S^2$ surrounding the origin:

$$\int_{S^2} \beta = 4\pi g,$$

as follows since $\oint_{S^2} \beta = g \, d\Omega$. But if $\beta$ were exact, $\beta = d\alpha$, then

$$\int_{S^2} \beta = \int_{S^2} d\alpha = \int_{S^2} \alpha = 0$$

since $S^2$ is a cycle ($\exists S^2 = 0$). So $\beta$ is not exact, and $[\beta] \neq 0$ defines an element of $H^2(M)$.

In ordinary language, $\beta = d\alpha$ (if it were true) would mean $\vec{B} = \nabla \times \vec{A}$. Since $\beta \neq d\alpha$, however, it means that there does not exist any $\vec{A}$ on $M$ such that $\vec{B} = \nabla \times \vec{A}$. At least, not any smooth $\vec{A}$. In books it is common to introduce a "monopole string", a line on which $\vec{A}$ is singular, with such singularities, you can have $\vec{A}$ such that $\vec{B} = \nabla \times \vec{A}$.

Since $[\beta] \neq 0$, we see that $H^2(M)$ is not trivial (it is not $\{0\}$). In fact, $H^2(M)$ is one-dimensional, and it is spanned by $[\beta]$. $H^2(M) = \mathbb{R}$, and every element of $H^2(M)$ can be written as $a[\beta]$, where $a \in \mathbb{R}$. Equivalently, if $\omega \in \Omega^2(M)$, then $\omega$ can be written,

$$\omega = a\beta + d\psi$$

for some $\psi \in \Omega^1(M)$. Interpreting $\omega$ as a magnetic field flux 2-form, we see that every smooth $\vec{B}$ on $M$ is the sum of a monopole field (of some strength) plus the curl of a smooth vector potential.
BTW, since $H^2(M) \cong \mathbb{R}$, by de Rham's thm we must have $H_2(M) = \mathbb{R}$, and there must be 2-cycles that are not boundaries.

Indeed there are: $S^2$ is one such.

Now we develop cohomology theory and its relation to topology.

First, a special case, $r=0$. We have

$$H^0(M) = \frac{Z^0(M)}{B^0(M)}.$$

0-forms on $M$ are scalar fields, $f: M \to \mathbb{R}$. An "exact 0-form" is a scalar field $f$ such that $f = d\alpha$ where $\alpha$ is a "(-1)-form". But (-1)-forms don't exist, so we understand that exact 0-forms don't exist either, except for $f=0$. That is, $B^0(M) = \{0\}$, so

$$H^0(M) = Z^0(M).$$

A closed 0-form is one satisfying $df = 0$. This means $f = \text{const.}$ on each connected component of $M$. If $M$ has $\nu$ connected components, then a closed 0-form on $M$ is specified by its const. values on each component.

$$(df = 0) \quad f = c_1 \quad f = c_2 \quad \ldots \quad f = c_\nu \quad \nu = \#\text{ connec. components}.$$

Thus, $H^0(M) = Z^0(M) \cong \mathbb{R}^\nu$. This is the same conclusion reached earlier with homology groups, giving us an instance of de Rham's theorem.
Now let's consider bases in \( H_r(M) \), \( H^r(M) \). Start with \( H_r(M) \).

Let us choose a basis in \( H_r(M) \). Each basis vector is an equivalence class of cycles, so choose one representative element in each class, call it \( e_i \), so that the basis \( \{ \} \) in \( H_r(M) \) is \( \{ [e_i] \} \) and \( e_i \in Z_r(M) \).

So an arbitrary element of \( H_r(M) \) can be written

\[
[z] = \sum_i c_i [e_i]
\]

where \( z \in Z_r(M) \) and \( c_i \in \mathbb{R} \) (the expansion coefficients). This means that

\[
z = \sum_i c_i \Xi e_i + \Xi c,
\]

where \( c \in C_{r+1}(M) \); thus, any cycle \( z \in Z_r(M) \) can be written in this form.

Similarly for \( H^r(M) \). Choose a basis in \( H^r(M) \). Each basis vector is an equivalence class of closed \( r \)-forms on \( M \). Pick representative elements, call them \( \{ \theta_i \} \), so that \( H^r(M) \) is spanned by \( \{ [\theta_i] \} \), and \( \theta_i \in Z^r(M) \), \( d\theta_i = 0 \). Now let \( \omega \) be any element of \( Z^r(M) \), i.e. \( d\omega = 0 \). Then \( [\omega] \in H^r(M) \) and

\[
[\omega] = \sum_i a_i [\theta_i]
\]

where \( a_i \in \mathbb{R} \). This means

\[
\omega = \sum_i a_i \theta_i + d\Psi
\]

for some \( \Psi \in \Omega^{r+1}(M) \). This is the general form for a closed \( r \)-form on \( M \).

So far this is a fairly trivial statement of the definition of a basis and of the quotient spaces \( H_r(M) \) and \( H^r(M) \). Now add de Rham's theorem. It tells us that the \( i' \)-sums above run from 1 to \( b_r(M) \) (the Betti...
number). It also tells us, that since $H^r(M) = H_r(M)^*$, we can choose the basis $\{\theta_i\}$ in $H^r(M)$ to be dual to the basis $\{e_i\}$ in $H_r(M)$. Let's do that, so that

$$([\theta_i], [e_j]) = (\theta_i, e_j) = \int_{e_j} \theta_i = \delta_{ij}.$$ 

Then the coefficients $a_i$ in the expansion of $\omega$ above can be computed by integrating $\omega$ over the basis cycles $\{e_i\}$. To see this just substitute,

$$(\omega, e_i) = \sum_j a_j (\theta_j, e_i) + (df, e_i)$$

because $df$ exact ei a cycle

$$= \sum_j a_j \delta_{ji} = a_i.$$ 

This leads to a theorem:

A closed form $\omega \in Z^r(M)$ is exact iff $\int_{e_i} \omega = 0$ for all $\{e_i\}$ in a basis in $H_r(M)$.

Example of basis: In monopole example above, let's take $S^2$ to be the basis 2-cycle, $e_1 = S^2$. Then $\theta_1$ can be taken to be $\beta/4\pi g$, so that $(\theta_1, e_1) = 1$.

Now for formal properties of cohomology groups. Begin with the ring of differential forms on $M$. This is the direct sum of all forms of all possible ranks,

$$\Omega(M) = \text{ring of diff. forms on } M$$

$$= \Omega^0(M) \oplus \cdots \oplus \Omega^m(M), \quad \text{where } m = \dim M.$$ 

An element of this ring is a (formal) linear combination of forms of different ranks, e.g., $dx \wedge dy + dz$. You can't integrate such
over surfaces (that requires a single rank), but such formal linear combinations are useful nonetheless. One reason for defining such an object is to obtain a set that is closed under the exterior product \( \wedge \). This is the definition of a ring, i.e., a set of objects that you can add or multiply, with the usual rules of distribution of addition over multiplication (and some other axioms). To say that \( \Omega(M) \) is a ring mainly conveys the idea that there is a multiplication law defined, \( \wedge \) in this case.

A single space like \( \Omega^r(M) \) is not a ring because it is not closed under \( \wedge \).

Similarly, we can define the cohomology ring,

\[
H^*(M) = H^0(M) \oplus \ldots \oplus H^m(M) = \text{cohomology ring.}
\]

Here the multiplication law is again \( \wedge \), but now defined on equivalence classes of closed forms (we have to make this definition). The obvious definition is (for \( \omega \in \Omega^r(M), \phi \in \Omega^s(M) \)):

\[
[\omega] \wedge [\phi] = [\omega \wedge \phi],
\]

but we must check this for consistency. First, note that \( \omega \wedge \phi \) is closed,

\[
d(\omega \wedge \phi) = d(\omega \wedge \phi) + (-1)^r \omega \wedge d\phi = 0
\]
since \( d\omega = d\phi = 0 \). Thus, \([\omega \wedge \phi]\) is meaningful as an element of \( H^{r+s}(M) \). Next, must show that the defn is indep. of the representative element in the equivalence class. Let \( \omega' = \omega + d\psi \).

Then

\[
[\omega' \wedge \phi] = [\omega \wedge \phi + d\psi \wedge \phi].
\]
But $d\psi \phi$ is exact,

$$d(\psi \phi) = d\psi \phi + (-1)^n \psi d\phi$$

so

$$[\omega \wedge \phi] = [\omega \phi + \text{exact}] = [\omega \phi].$$

Similarly, if you write $\phi' = \phi + d\chi$, altogether, we have shown that

$$[\omega] \wedge [\eta] = [\omega \eta]$$

is a consistent definition, and hence $H^*(M)$ is a ring under $\wedge$.

Now let's study behavior of cohomology groups and rings under maps. Let $f : M \to N$ be a smooth map between manifolds. We know how to pull back forms, i.e., if $\omega \in \Omega^r(N)$ then $f^*\omega \in \Omega^r(M)$.

What about cohomology group elements? Can we pull them back? Let $\omega \in Z^r(M)$, $d\omega = 0$, and let's try to define $f^*[\omega]$ by

$$f^*[\omega] = [f^*\omega]$$

(The obvious definition). But we must check consistency. First, $d(f^*\omega) = f^*(d\omega) = 0$ since $d\omega = 0$, so $[f^*\omega]$ is meaningful as an element of $H^r(M)$. Next, if $\omega = \omega + d\psi$, then

$$f^*[\omega'] = [f^*(\omega + d\psi)] = [f^*\omega + f^*d\psi] = [f^*\omega + d(f^*\psi)]$$

$$= [f^*\omega].$$
So the result is indep. of the rep. element chosen in \([\omega]\), and the definition works. We have defined a new meaning to \(f^*\):

\[
f^* : \Omega^r(N) \rightarrow \Omega^r(M) \quad \text{(old meaning)}
\]

\[
f^* : H^r(N) \rightarrow H^r(M) \quad \text{(new meaning)}
\]

\(f^*\) is a linear map of cohomology groups.

\(f^*\) also preserves the \(\wedge\) between cohomology group elements, if we as we see by pursuing the defn. of \(\wedge\) and \(f^*\) on such things:

\[
f^*([\omega] \wedge [\eta]) = f^*([\omega \wedge \eta]) = f^*(\omega \wedge \eta)
\]

\[
= (f^*[\omega]) \wedge (f^*[\eta]) = [f^*[\omega]] \wedge [f^*[\eta]]
\]

So, you can take \(\wedge\) first and then apply \(f^*\), or do it in the reverse order, answers are the same. This means that \(f^* : H^*(N) \rightarrow H^*(M)\) is a ring homomorphism (another way of stating same thing).

---

Will need the following result concerning the behavior of integrals under maps in the subsequent discussion of potentials. Let \(f : M \rightarrow N\) be a map between manifolds, let \(c \in C^r(M)\) be an \(r\)-chain on \(M\), let \(\omega \in \Omega^r(N)\):

![Diagram](image)

The map \(f\) can be used to push \(c\) forward to \(N\), where it becomes