Next we consider induced vector fields, which you have when you have an action of a Lie group on a manifold $M$. First the intuitive picture.

Consider a vector $V \in \mathfrak{g}$. Intuitively its base is the identity $e$ and its tip is a nearby (near-identity) group element, call it $\Phi_e = \exp(\epsilon)$. The map $\Phi_e = \text{id}_M$ does nothing to points of $M$, but $\Phi_e$ causes the points of $M$ to get up and move a short distance, creating a vector field on $M$. In this way we associate $V \in \mathfrak{g}$ with a vector field $V_M \in \mathfrak{X}(M)$ ($V_M$ is denoted $V^\#$ by Nakahara.)

To make this more precise, we replace $\epsilon$ by $\sigma(t) = \Phi_{e^t}$, $e = \exp(\epsilon V)$ for small $t$, using earlier notation for integral curves on the group manifold, and consider the action of $\Phi$ on $M$. To make this more precise we need to talk about advance maps on the group manifold, earlier denoted $\Phi_{v, t}$, and the action of $G$ on $M$, which will be denoted $g \mapsto \Phi^g$. To avoid confusion, let's use $\Psi_{v, t}$ for the advance map on $G$, $\Phi^g$ for the action of $G$ on $M$.

When $t$ is small, $\Psi_{v, t} e = \exp(tV)$ is close to $e$, so we can identify it with $e$ above, in the picture. When acting on $x \in M$, $\Phi_e = \Phi_{\exp(tV)}$ causes $x$ to move to a nearby point,
thereby making a small vector on \( M \). Letting this vector act on a function \( f: M \to \mathbb{R} \), we have

\[
    f(\Phi_{\exp(tv)} x) - f(x)
\]

\[
= (\Phi^*_{\exp(tv)} f)(x) - f(x)
\]

\[
= ((\Phi^*_{\exp(tv)} - 1)f)(x), \quad \text{(think t small)}.
\]

Suggests we define \( V_M \in \mathfrak{X}(M) \) by

\[
(V_M f)(x) = \left( \left. \frac{d}{dt} \right|_{t=0} \Phi^*_{\exp(tv)} f \right)(x),
\]

or, since \( x \) and \( f \) are arbitrary,

\[
V_M = \left. \frac{d}{dt} \right|_{t=0} \Phi^*_{\exp(tv)}
\]

(Both sides understood as operators: \( \mathcal{F}(M) \to \mathcal{F}(M) \).

See Nakahara Eq. 5.160. He drops the star on \( \Phi \) and just writes \( g \) instead of \( \Phi_g \), where here \( g = \exp(tv) \).

\( V_M \) is called the induced vector field. It is also called the infinitesimal generator of the action \( g \mapsto \Phi_g \).
An equivalent way to define the induced vector field $V_M$ eval.
at a point $x \in M$ is an equivalence class of curves. One of these
curves is easy to write down. Let $c : \mathbb{R} \to M$ be defined by

$$c(t) = \Phi_{\text{exp}(tV)} x.$$  

Then $c(0) = x$, and $[c] = V_M|_x$.

An application of induced vector fields. Let $M$ be the configuration
space of a mechanical system. Suppose a chart with coordinates $q^\mu$.
The Lagrangian is a function $L(q, \dot{q})$. Let $G$ be a group with
an action $g \mapsto \Phi_g$ on $M$, and suppose that $L$ is invariant under
the group action. This means that $\Phi^*_g L = L$, $\forall g \in G$. (But
we won't define what $\Phi^*_g$ means here, just say that there is an
obvious definition.) Then for every $V \in \mathfrak{g}$ there is a conserved
quantity $C_V$, where

$$C_V = (p, V_M) = \Phi^*_\mu (V_M)^\mu, \quad \frac{dC_V}{dt} = 0.$$  

Here $p_\mu = \frac{\partial L}{\partial \dot{q}^\mu}$ is the canonical momentum. This is
Noether's theorem.
Now an application of induced vector fields and an intro to geometric mechanics, namely, Noether's theorem. Let $\mathbb{M}$ be the configuration manifold of a mechanical system, and let $x^i$ be coordinates on $\mathbb{M}$. They need not be rectangular coordinates. The Lagrangian is a function $L(x^i, \dot{x}^i)$. It is not a function on $\mathbb{M}$, but rather on $T\mathbb{M}$. A point of $T\mathbb{M}$ is a tangent vector $\nu$ attached to a point $x \in \mathbb{M}$. If $x^i$ are the coordinates of $x$ and $\nu = \dot{x}^i \partial / \partial x^i$, then $x^i, \dot{x}^i$ are coordinates on $T\mathbb{M}$, and the Lagrangian is usually expressed in terms of these coordinates. Let's henceforth use $\nu^i$ instead of $\dot{x}^i$ for coordinates on the fiber $T_x \mathbb{M}$, since it's less confusing. (Or rather, we'll use $\dot{x}^i$ or $v^i$, whichever seems less confusing.)

Thus

$$\nu = \nu^i \partial / \partial x^i$$

and

$$L : T\mathbb{M} \to \mathbb{R}.$$ 

Here's a picture of the tangent bundle and some fibers:

The bundle has the usual projection,

$$\pi : T\mathbb{M} \to \mathbb{M} : (x, \nu) \mapsto x,$$

which just throws away the velocity information. Notice that

$$\pi^{-1}(x) = T_x \mathbb{M} = \text{the fiber over } x.$$
Now suppose we have a group $G$ with an action $g \mapsto \Phi_g$ on $M$, so that $\Phi_g \Phi_h = \Phi_{gh}$, and $\Phi_g : M \to N$. Let $V \in \mathfrak{g}$, and let $\exp(\lambda V)$ be the corresponding 1-parameter subgroup of $G$, with $\lambda$ as a parameter. Then the corresponding induced vector field on $M$ is \( \xi \) (since we will use $t$ for "real" time.)

\[
V_M = \frac{d}{d\lambda}\big|_{\lambda=0} \Phi_{\exp(\lambda V)} \text{ acting on } \mathfrak{g}(M),
\]

or as an equivalence class of curves through $x \in M$,

\[
V_M \big|_x = [\Phi_{\exp(\lambda V)} x],
\]

i.e., $c(\lambda) = \Phi_{\exp(\lambda V)} x$, $c : \mathbb{R} \to M$.

Noether's theorem applies when the Lagrangian is invariant under the group action (this is the simplest version of the theorem). But the Lagrangian is not a function on $M$, rather it's on $TM$, so we have to say what this means. This is easy in coordinates. Let

\[
y = \Phi_{\exp(\lambda V)} x,
\]

or, in coordinates,

\[
y^i = \Phi^i(\lambda, x) \quad (V \text{ undetermined}). \quad (A)
\]

Then we will say that $L$ is invariant under the group action if

\[
L(x^i, \dot{x}^i) = L(y^i, \dot{y}^i), \quad (B)
\]

where $y^i$ is a function of $x$ and $\lambda$ as in (A), and where $y^i$ means

\[
y^i = \frac{\partial y^i}{\partial x^j} \dot{x}^j = \frac{\partial \Phi^i(\lambda, x)}{\partial x^j} \dot{x}^j \quad (c)
\]
Eq. (c) is the obvious application of the chain rule to (A). Note that the induced vector field has components,

\[(V_M)^i = \frac{\partial \Phi^i}{\partial \lambda} \bigg|_{\lambda = 0}, \tag{D}\]

because that is the rate of change of the coordinates wrt \(\lambda\) when the group action \(\hat{\Phi}_\lambda(x)\) is applied to \(x \in M\). Now Eq. (B) is supposed to hold for all \(\lambda\), but the LHS doesn't depend on \(\lambda\), so if we apply \(d/d\lambda\) we get

\[0 = \frac{\partial L}{\partial y^i} \frac{\partial}{\partial \lambda} \Phi^i + \frac{\partial L}{\partial y^i} \frac{\partial^2 \Phi^i}{\partial x \partial y^j} \dot{x}^j.\]

Now setting \(\lambda = 0\), where \(\dot{x}^i = \dot{y}^i\), \(\ddot{x}^i = \ddot{y}^i\), and (D) holds, we get

\[0 = \frac{\partial L}{\partial x^i} (V_m)^i + \frac{\partial L}{\partial \dot{x}^i} \frac{\partial (V_m)^i}{\dot{x}^j}.\]

But by Euler-Lagrange equations,

\[\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) = \frac{\partial L}{\partial x^i},\]

this gives

\[0 = \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{x}^i} \right) (V_m)^i + \frac{\partial L}{\partial \dot{x}^i} \frac{d}{dt} (V_m)^i\]

\[\tag{E}\]

\[\frac{d}{dt} \left( \frac{\partial L}{\partial x^i} V_m^i \right) = 0.\]

The quantity \(\partial L/\partial \dot{x}^i = p_i\) is the momentum conjugate to \(x^i\), so we have

\[\frac{d}{dt} \left( p_i V_m^i \right) = 0.\]

The conserved quantity is the momentum contracted with the
induced vector field of the group action. This is Noether's theorem. It provides a map from $\mathfrak{g}$ to conserved quantities, that is scalars on $TM$ that are invariant under the time evolution.

Now for some geometrical interpretation. We started with an action of $G$ on $M$, $g \mapsto \Phi_g$, $\Phi_g : M \to M$. But to talk about the invariance of $L$ under the group action, we need an action of $G$ on $TM$. The key formula is (c), which shows how the velocity should transform. The Jacobian $\partial y^i / \partial x^j$ in this formula shows that the new velocity $y^i$ is the tangent map of $\Phi_g$ applied to the old velocity. To draw a picture, let $y = \Phi_g x$ and let $v \in T_x M$.

Let $v = \dot{x}^i \partial / \partial x^i |_x$ and $w = \dot{y}^j \partial / \partial x^j |_y$. Then $w = \Phi_g * x |_x v$ is equivalent to (c). What we have obtained is a lifted action of $G$ on $TM$, call it $g \mapsto \tilde{\Phi}_g$, $\tilde{\Phi}_g : TM \to TM$, associated with $\Phi_g$ on $M$.

$$\tilde{\Phi}_g (x, v) = (\Phi_g x, \Phi_g * e v),$$

where $v \in T_x M$. Then (B) is equivalent to

$$L(x, v) = L(\tilde{\Phi}_g (x, v)) = (\tilde{\Phi}_g * L)(x, v),$$

or

$$\tilde{\Phi}_g^* L = L.$$
The Lagrangian is invariant under the lifted action. If we now simply set \( g = \exp(\lambda V) \) and apply \( \frac{\partial}{\partial \lambda} \bigg|_{\lambda=0} \), we get

\[
\mathcal{L} = \mathcal{L}_{\nu_{TM}} L = 0,
\]

where \( \nu_{TM} \) is the induced vector field on \( TM \) associated with the action \( \mathcal{L} = \exp(\lambda V) \), and \( \mathcal{L} \) is the Lie derivative.

The final result \( \mathcal{E} \) for the conserved quantity is obviously the contraction of a 1-form with components \( \nu_i \) with the induced vector field \( \nu_{TM} \) of the symmetry. The momentum \( \mathcal{E} \) appears as the components of a 1-form. But it is not a field of forms over \( M \), that is, it is not a member of \( \Omega^1(M) \), because \( \nu_i = \partial \mathcal{E}/\partial x^i \) is a function of both \( x^i \) and \( \xi^i \). That is, it is an association of each point of \( TM \) with a specific covector. This is usually regarded as a map from the tangent bundle to the cotangent bundle.

\[
F : TM \to T^*M \\
\mathcal{E} : (x, v) \to (x, \mathcal{E})
\]

where

\[
\nu = \nu_i \frac{\partial}{\partial x^i}
\]

\[
\mathcal{E} = \mathcal{E}_i \frac{\partial}{\partial x^i} \bigg|_{x}
\]

This map \( F \) is essentially the Legendre transformation of mechanics. It preserves fibers, so it can also be thought of as a map: \( T_xM \to T^*_xM \) (\( x \) does not change under the map).
Just looking at what $F: T\mathcal{M} \to T^*\mathcal{M}$ does on a single fiber, the equation specifying $F$ is just the usual definition of the momentum,

$$p_i = \frac{\partial L}{\partial \dot{x}_i}(x, \dot{x}).$$

Thus, the map is invertible if the Jacobian

$$\frac{\partial p_i}{\partial v_j} = \frac{\partial^2 L}{\partial v_i \partial v_j}$$

has full rank everywhere, i.e., if its determinant is $\neq 0$.

Lagrangians for which

$$\det \left( \frac{\partial^2 L}{\partial v_i \partial v_j} \right) \neq 0$$

everywhere on $TM$ are said to be regular. For a regular Lagrangian, $F: T\mathcal{M} \to T^*\mathcal{M}$ is a diffeomorphism, which can be used to push-forward any geometrical structure (e.g., the time flow) from $TM$ to $T^*\mathcal{M}$, or conversely to pull anything back from $T^*\mathcal{M}$ to $TM$.

In particular, the Hamiltonian, defined as

$$H = p_i \dot{x}_i - L(x, \dot{x})$$

begins life as a function on $TM$, but pushed forward under $F$ to $T^*\mathcal{M}$ it becomes a function of $(x, p)$. Then one can show that the equations of motion on $T^*\mathcal{M}$ are Hamilton's equations,

$$\begin{align*}
\dot{x}_i &= \frac{\partial H}{\partial p_i} \\
\dot{p}_i &= -\frac{\partial H}{\partial x_i}
\end{align*}$$
This is a vector field on \( T^* N \), that is, a section of \( T(T^* N) \).

This is the beginning of geometrical mechanics, about which we will say more later.

The Lagrangians of elementary, non-relativistic mechanics are almost always regular, but relativistic mechanics and field theory (and general relativity) lead to (usually) irregular Lagrangians. More about that later.