Recall, a tensor of type \( (r,s) \) at a point \( p \in M \) is a multilinear map,

\[
T: \underbrace{T^*_p M \times \cdots \times T^*_p M}_r \times \underbrace{T_p M \times \cdots \times T_p M}_s \to \mathbb{R}.
\]

The components of \( T \) are given by (w.r.t. a chart)

\[
T^{i_1 \ldots i_r}_{j_1 \ldots j_s} = T(\mathrm{d}x^{i_1}, \ldots, \mathrm{d}x^{i_r}; \frac{\partial}{\partial x^{j_1}}, \ldots, \frac{\partial}{\partial x^{j_s}}).
\]

A tensor field on \( M \) is an assignment of a tensor at each point \( p \in M \). The components of a tensor field are functions of position.

Now we consider the behavior of fields under maps. Let \( f: M \to N \) be a map between manifolds, let \( p \in M \) and \( q \in N \) be points such that \( q = f(p) \).

\[ \begin{array}{c}
\text{M} \\
\text{\textbullet} \quad p \\
\text{\textbullet} \quad f \\
\text{\textbullet} \quad q \\
\text{\textbullet} \quad \text{N}
\end{array} \]

\( \text{dim} \, M = m \)

\( \text{dim} \, N = n \)

\( \text{Don't have to have same dimensionality} \)

**Question:** Given \( X \in T_p M \), is there any way to associate it with a \( Y \in T_q N \)? Yes, use the small displacement interpretation of a tangent vector (at close to \( p \)), and define \( q_1 = f(p_1) \).

(You map both the base and the tip of the small arrow under \( f \),...
to get a new small arrow on \( N \). Both are understood to be displacements taking place in some elapsed parameter \( \Delta t \).}

Rather, in the equivalence class of curves interpretation, just map the curves themselves:

Thus we can say, if \( X = [c] \), then \( Y = [foc] \). This defines a map

\[
\hat{f}_* : T_p M \rightarrow T_q N
\]

where \( \hat{f}_* \) is called the tangent map, derivative map, or push-forward. Note that \( \hat{f}_* \) can be defined succinctly by

\[
\hat{f}_* [c] = [foc].
\]

If we impose coordinates (charts) on \( M \) and \( N \) (containing \( p \) and \( q \)), we can write \( \hat{f}_* \) in coordinates. Let \( x^i \) be coordinates on \( M \) and \( y^i \) those on \( M \). Let

\[
X = \sum_{i=1}^{m} x^i \frac{\partial}{\partial x^i}_p
\]

\[
\hat{f}_* X = Y = \sum_{i=1}^{m} y^i \frac{\partial}{\partial y^i}_q
\]
Then
\[ Y^i = \sum_j \frac{\partial y^i}{\partial x^j} X^j, \]

where \( \frac{\partial y^i}{\partial x^j} \) is the coordinate representation of the derivatives of \( f \).

This is how we map vectors. For covectors, it works the other way, i.e., given a covector \( \alpha \in T^*_p N \), we can associate it with another covector \( \beta \in T^*_p M \).

That is, we define a map \( f^*: T^*_q N \to T^*_p M \) by demanding
\[ \beta(Y) = \alpha(Y) \text{ when } Y = f_\# X. \]
That is, \( \beta = f^* \alpha \) is defined by
\[ (f^* \alpha)(X) = \alpha(f_\# X), \quad \forall \ X \in T^*_p M. \]

The covector \( f^* \alpha \in T^*_p M \) is said to be the pull-back of \( \alpha \in T^*_q N \), because it works in the opposite direction to \( f \).

This is for a covector at a point (two of them, \( \alpha \) and \( f^* \alpha \)).

If now we let \( \alpha \) be a covector field (same symbol, new meaning), \( \alpha \in \mathfrak{X}^*(N) \), then \( f^* \alpha \in \mathfrak{X}^*(M) \), is given by
\((f^*\alpha)_p(x) = \alpha_{r(p)}(f^*x), \quad \forall \ p \in M\)

where the subscript indicates the point at which the field is evaluated.

Thus 1-forms on \(\mathbb{R}\) get pulled back into 1-forms on \(M\).

A simpler example of the pull-back is for scalar fields. Let \(\phi \in \mathcal{F}(\mathbb{R}), \phi: \mathbb{R} \to \mathbb{R}\) be a scalar field. Then we define the pull back \(\psi = f^*\phi \in \mathcal{F}(\mathbb{R})\) by

\[\psi(p) = (f^*\phi)(p) = \phi(f(p)) = (\phi \circ f)(p),\]

that is, \(f^*\phi = \phi \circ f\).

Return to the pull-back of vectors, and put it into coordinate language. Let \(x^1, \ldots, x^n\) be coordinates on \(\mathbb{R}^n\) as above, and let \(\beta = f^*\alpha\), so that

\[\alpha = \sum_{i=1}^{n} \alpha_i \, dy_i|_p\]

\[\beta = \sum_{i=1}^{m} \beta_i \, dx_i|_p\]

Then

\[\beta_i = \sum_{j=1}^{n} \frac{\partial y_j}{\partial x_i} \alpha_j.\]

Notice that \(f^{-1}\) need not be defined in order to define \(f_*\) and \(f^*\). In particular, \(M\) and \(\mathbb{R}\) need not have the same dimensionality.

But if \(f^{-1}\) does exist (then \(\dim M = \dim \mathbb{R}\)), then we can "push forward" vectors from \(M\) to \(\mathbb{R}\) (by \(f^{-1}*)\) and

(continued page after next)
Behavior of $f^*$, $f_*$ under compositions. Let $f: M \rightarrow N$,

Then $g \circ f: M \rightarrow P$,

and $(g \circ f)_* = g_* \circ f_*$.  

This is fairly obvious, you just map the small displacement vector in $M$ under a succession of 2 linear maps, first by $f_*$, then $g_*$, to get $(g \circ f)_*$.  

As for pull-backs, the rule is 

$$(g \circ f)^* = f^* \circ g^*$$  (in reverse order).
"pull-back" vectors from \( N \) to \( M \) (by \( f_* \)).

Now consider the mapping of one manifold of a certain dimensionality into one of higher dimensionality, \( f: M \to N \), \( \dim M \leq \dim N \). Then \( f \) is called an immersion if \( f_* \) is of maximal rank,

\[
\text{rank } f_* : T_p M \to T_{f(p)} N = \dim M.
\]

This means that each little piece of \( M \), which looks like \( \mathbb{R}^m \), gets mapped into a subset of \( N \) that also looks like \( \mathbb{R}^m \). This means that \( f_* \) is injective (the image of \( M \) under \( f \) is locally \( m \)-dimensional).

However, an immersion does not preclude self-intersections:

To exclude self-intersections, we can demand that \( f \) itself be an injection. This means then \( f \) is called an embedding (because \( \text{im } f \) "looks like" \( M \)).

Now we consider ordinary differential equations (ODE's) and flows. Begin with an intuitive picture of a vector field, as a small displacement (each understood to be taking place in some elapsed parametric \( \Delta t \)), attached to each point of \( M \):
If you just follow these arrows, starting with some initial point \( x_0 \), you trace out a curve called the integral curve of \( X \). By following this integral curve for time \( t \), starting at \( x_0 \), you get a final point described by a function

\[
\Phi : M \times \mathbb{R} \rightarrow M,
\]

\[
x = \Phi(x_0, t),
\]

where \( \Phi \) is called \underline{the advance map}.

To make this more precise, express the vector field \( X \) in some chart:

\[
X = \sum_i X^i(x) \frac{\partial}{\partial x^i}
\]

This is an operator which when acting on scalars \( f \) gives a number interpreted as \( df/\,dt \). In particular, letting it act on the coordinates themselves gives a set of ODEs:

\[
\frac{dx^i}{dt} = X^i(x).
\]

Thus, a vector field on a manifold is a generalization of a system of ODE's on \( \mathbb{R}^n \). Standard theorems on ODE's say that the system above has a unique solution \( x^i(t) \) satisfying \( x^i(0) = x^i_0 \) for \( t \) in some interval.
[0,1], if the functions $X^i(x)$ are smooth. This is the (important) uniqueness theorem for ODE's (really, existence and uniqueness).

However, even if the vector field $X^i(x)$ is smooth, the solution may not exist for all $t$ (for example, it may run off to infinity in finite $t$). (For an example of this, consider $\dot{x} = x^2$, $x_0 = 1$ ($x \in \mathbb{R}$), for which $x \to \infty$ as $t \to 1$.) For simplicity, we will assume that this does not happen, i.e., that solutions $x^i(t)$ exist for all time, for any $x_0$. Then we can speak of the "general solution functions" $\Phi^i(t, x_0)$ that give $x^i(t)$, assuming $\dot{x}_0 = x_0^i$. These solution functions satisfy,

1) \[ \Phi^i(0, x_0) = x_0^i \]

2) \[ \frac{d\Phi^i}{dt}(t, x_0) = X^i(\Phi(t, x_0)). \]

These are just the initial conditions and ODE's expressed in terms of $\Phi^i$. [We would normally write them,

1) \[ x^i(0) = x_0^i \]

2) \[ \frac{dx^i}{dt} = X^i(x(t)). \]

All of the above is in one chart. By mapping a solution $x^i(t)$ in the given chart back onto $M$, we get a segment of an integral curve. Just before we run off one chart we can switch to another, thereby continuing the integral curve. The result is that we define a map $\Phi : \mathbb{R} \times M \to M$ or maps $\Phi_i : M \to M$ (a different notation), such that