Summary

(M only)

Basis vectors \( \{ e_\mu \} \)

Basis forms \( \{ \theta^\mu \} \)

Dual bases, \( \theta^\mu(e_\nu) = \delta^\mu_\nu \)

Structure constant, \( [e_\mu, e_\nu] = c^\sigma_{\mu\nu} e_\sigma \)

or \( d\theta^\mu = -\frac{1}{2} c^\mu_{\alpha\beta} \theta^\alpha \wedge \theta^\beta \)

Comma notation: \( e_\mu f = f,_{\mu} \) for \( f : M \to \mathbb{R} \)

Now \((M, \nabla)\) (but no \(g\), no assumptions on \(\nabla\))

\( \nabla_\mu \equiv \nabla_{e_\mu} \)

\( \nabla_\mu e_\nu = \Gamma_{\mu\nu}^\alpha e_\alpha \) (defn of \( \Gamma_{\mu\nu}^\alpha \))

\( \nabla_\mu \theta^\alpha = -\Gamma_{\mu\nu}^\alpha \theta^\nu \) (from \( \theta^\mu(e_\nu) = \delta^\mu_\nu \))

\( T(x, y) = \nabla_x y - \nabla_y x - [x, y] \)

\( T(e_\alpha, e_\beta) = T_{\alpha\beta}^\mu e_\mu \) (defn of \( T_{\alpha\beta}^\mu \))

\( T_{\alpha\beta}^\mu = \Gamma_{\alpha\beta}^\mu - \Gamma_{\beta\alpha}^\mu - c_{\alpha\beta}^\mu \)

\( R(x, y) = [\nabla_x, \nabla_y] - \nabla_{[x, y]} \)

\( R(e_\alpha, e_\beta) e_\nu = R_{\nu\alpha\beta}^\mu e_\mu \) (defn of \( R_{\nu\alpha\beta}^\mu \))

\( R_{\nu\alpha\beta}^\mu = \Gamma_{\beta\alpha,\nu}^\mu - \Gamma_{\alpha\beta}^\mu,_{\nu} + \Gamma_{\alpha\beta\nu}^\sigma \Gamma_{\nu\sigma}^\mu - \Gamma_{\beta\alpha\nu}^\sigma \Gamma_{\nu\sigma}^\mu - c_{\nu\beta}^\sigma \Gamma_{\alpha\nu}^\sigma \)
Cartan's definition:

\[ \omega_{\mu \nu} = \Gamma_{\alpha \nu}^\mu \Theta^\alpha \]  
(\text{Lie-alg.-valued 1-form})

\[ T^\mu = \frac{1}{2} T_{\alpha \beta}^\mu \Theta^{\alpha} \wedge \Theta^{\beta} \]  
(vector-valued 2-form)

\[ R_{\mu \nu}^\alpha = \frac{1}{2} R_{\nu \alpha \beta}^\mu \Theta^{\alpha} \wedge \Theta^{\beta} \]  
(Lie alg.-valued 2-form)

\[ d\Theta^\mu + \omega_{\mu \nu} \wedge \Theta^\nu = T^\mu \]  
1st Cartan structure.

The LHS can be interpreted as a "covariant exterior derivative". This eqn. can be interpreted as an alternative definition of the torsion (equivalent to the usual one by the calculation above).
\[
\begin{align*}
\frac{1}{2} \left( \Gamma^\mu_{\nu,\alpha} - \Gamma^\mu_{\alpha,\nu} - \Gamma^\mu_{\sigma,\nu} \right) \theta^\alpha \wedge \theta^\beta \\
= \frac{1}{2} \left( R^\mu_{\nu,\alpha \beta} - \Gamma^\mu_{\beta,\nu} \Gamma^\alpha_{\sigma} + \Gamma^\alpha_{\nu} \Gamma^\mu_{\beta,\sigma} \right) \theta^\alpha \wedge \theta^\beta \\
= R^\mu_{\nu} - \frac{1}{2} \omega^\mu_{\sigma} \wedge \omega^\sigma_{\nu} + \frac{1}{2} \omega^\nu_{\sigma} \wedge \omega^\mu_{\sigma},
\end{align*}
\]

\[
\nabla \text{equal}
\]

\[
\begin{array}{c}
\alpha, \\
d \omega^\mu_{\nu} + \omega^\mu_{\sigma} \wedge \omega^\sigma_{\nu} = R^\mu_{\nu}
\end{array}
\]

2nd Cartan structure eqn.

Again, take

\[
T^\mu = d\theta^\mu + \omega^\mu_{\alpha} \wedge \theta^\alpha \]

apply \( d \),

\[
d T^\mu = 0 + \left( R^\mu_{\nu,\alpha} - \omega^\mu_{\sigma} \wedge \omega^\sigma_{\alpha} \right) \wedge \theta^\alpha
\]

\[
- \omega^\mu_{\alpha} \wedge \theta^\alpha \rightarrow T^\alpha = \omega^\mu_{\beta} \wedge \theta^\beta
\]

\[
\begin{array}{c}
d T^\mu + \omega^\mu_{\alpha} \wedge T^\alpha = R^\mu_{\alpha} \wedge \theta^\alpha
\end{array}
\]

1st Bianchi identity, generalized to case \( T \neq 0 \).

Finally, take 2nd Cartan structure, apply \( d \):

\[
d R^\mu_{\nu} = d \omega^\mu_{\sigma} \wedge \omega^\sigma_{\nu} - \omega^\mu_{\sigma} \wedge d \omega^\sigma_{\nu}
\]

\[
= \left( R^\mu_{\sigma} - \omega^\mu_{\alpha} \wedge \omega^\sigma_{\alpha} \right) \wedge \omega^\sigma_{\nu}
\]

\[
- \omega^\mu_{\sigma} \wedge \left( R^\sigma_{\nu} - \omega^\sigma_{\alpha} \wedge \omega^\sigma_{\nu} \right)
\]
One might say that the covariant exterior derivative of the curvature 2-form is 0, that this form is closed in this sense.

\[ dR^\lambda_{\mu} + \omega^\lambda_{\mu \nu} \wedge R^\nu_{\nu} - R^\lambda_{\mu \nu} \wedge \omega^\nu_{\nu} = 0 \]

This is the 2nd Bianchi identity, generalized.

When \( T = 0 \), these forms should reduce to the previous versions of the Bianchi identities. For the 1st Bianchi identity, this gives

\[ 0 = R^\lambda_{\mu \nu} \wedge \theta^\nu = \frac{1}{2} R^\lambda_{\mu \nu \beta} \theta^\nu \wedge \theta^\mu \wedge \theta^\beta \]

\[ \Rightarrow R^\lambda_{\mu \nu \beta} \equiv 0. \quad \text{Checks.} \]

For the 2nd Bianchi identity, notice that it doesn't involve \( T \) at all. But if you want to show equivalence to \( R^\lambda_{\mu \nu} [\alpha \beta ; \gamma] = 0 \), you must use \( T = 0 \).

Now consider the case that we have a metric \( g \) and a metric connection \( \nabla g = 0 \).

Then it is convenient to assume the basis \( \{ e_\alpha \} \) is orthonormal, i.e.,

\[ g_{\alpha \beta} = g (e_\alpha, e_\beta) = \eta_{\alpha \beta} \quad \text{(pseudo-Riem. case, or } \delta_{\alpha \beta}, \text{ Riem. case).} \]

\[ = \text{const. metric of special relativity.} \]

We know that if the curvature tensor \( \neq 0 \), then there is no coordinate basis such that \( g_{\alpha \beta} = \eta_{\alpha \beta} \). But there are always non-coordinate bases that make this true. This is a special kind of vielbein.
Thus are some special properties of $\Gamma$, $R$ in orthonormal vielbeins. First, $\nabla g = 0$ implies

$$0 = g_{\mu\nu,\alpha} - \Gamma_{\mu\nu}^\beta g_{\rho\nu} - \Gamma_{\nu\rho}^\beta g_{\mu\rho}. $$

Define $\Gamma_{\mu\nu} = g_{\rho\sigma} \Gamma_{\mu\nu}^{\beta}$. Note, this $\Gamma_{\mu\nu}$ is the 1-form index.

Also, in an orthonormal vielbein, $g_{\mu\nu} = \eta_{\mu\nu}$ so $g_{\mu\nu,\alpha} = 0$. Thus,

$$\Gamma_{\mu\nu\omega} + \Gamma_{\nu\omega\mu} = 0,$$

and $\Gamma_{\mu\nu}$ is antisymmetric in $\mu\nu$. (Recall in coord. basis $\omega$. LC connection, $\Gamma_{\mu\nu} = \Gamma_{\nu\mu}$) This property depends only on $\nabla g = 0$ (the parallel transport proceed by orthogonal (or Lorentz) transformations), it does not require the LC connection.

In terms of Cartan's forms, this condition is

$$\omega_{\mu\nu} = -\omega_{\nu\mu} \quad (\omega_{\mu\nu} = \eta_{\mu\nu}\omega_{\nu}. )$$

Similarly, we have

$$R_{\mu\nu} = -R_{\nu\mu} \quad (R_{\mu\nu} = \eta_{\mu\nu} R_{\rho\sigma}^{\rho\sigma}, \text{ Riemann-Cartan})$$

for the same reason.

Also note, if in addition $T = 0$ (LC connection) then

$$\Gamma_{\mu\nu}^{\sigma} = -\frac{1}{2} \left( C_{\mu\nu}^{\sigma} + C_{\nu\rho}^{\sigma} + C_{\rho\mu}^{\sigma}\right)$$

Now we consider a change of basis for an orthonormal vielbein.

To be specific, we'll assume the pseudo-Riemannian $(1+3)$ case, with $g_{\mu\nu} = \eta_{\mu\nu}$. A change of basis maps one orthonormal vielbein to another.

We are assuming that $g(e_\mu, e_\nu) = \eta_{\mu\nu}$.

Let $e'_\mu = \Lambda_\mu^\beta e_\beta$, and demand that $g(e'_\mu, e'_\nu) = \eta_{\mu\nu}$,

so the new vielbein is also orthonormal.
Let \( e'_\alpha = \Lambda^\alpha_\beta e_\beta \), define \( \Lambda^\alpha_\beta \). Then define

\[ g(e'_\alpha, e'_\beta) = \eta_{\alpha\beta}, \text{ and you find} \]

\[ \Lambda^\alpha_\mu \eta_{\alpha\beta} \Lambda^\beta_\nu = \eta_{\mu\nu} \]

where indices are raised and lowered with \( \eta \). Thus \( \Lambda^\alpha_\mu(x) \) is an \( x \)-dependent Lorentz transformation. These are gauge transformations in GR. How other things transform:

\[ \theta'^\mu = \Lambda^\mu_\alpha \theta^\alpha. \]

Any tensor transforms pointwise-linearly in \( \Lambda(x) \), for example, the Riemann–Cartan 2-form,

\[ R'^\alpha_\mu = \Lambda^\alpha_\alpha \Lambda^\beta_\nu R^\alpha_\beta. \]

But the Cartan–Connection 1-form has a less simple transformation law (since \( \Gamma \) is not a tensor):

\[ \omega'^\gamma_\nu = \Lambda^\gamma\gamma \Lambda^\beta_\nu \omega^\alpha_\beta - \Lambda^\gamma_\nu (\Lambda^\alpha_\gamma) \theta^\alpha. \]

The extra term on the right is characteristic of the transformation laws for gauge potentials.
Now we deal with the variational formulation of GR. We work in coordinates $x^\mu$. We start with the vacuum (matter-free) case, for which the field equs are

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0.$$ 

We seek a Lagrangian density $L_\phi$ such that these equs follow from

$$\delta \int d^4x \sqrt{-g} \ L_\phi = 0.$$ 

Here $d^4x = dx^0 \ldots dx^3$, $g = \det g_{\mu\nu} < 0$, so $-g = |g|$. The product $d^4x \sqrt{-g}$ is the invariant volume element, as will be explained later in the course. $L_\phi$ must be a scalar in order that the integral be independent of coordinates. The simplest scalar that can be constructed out of $g_{\mu\nu}$ and its derivatives (apart from trivial things like $g^{\mu\nu} = 1$) is the curvature scalar $R$. So we guess that $L_\phi \propto R$, and we look at the variation,

$$\delta\int d^4x \sqrt{-g} R = 0.$$ 

The variation is carried out by $g_{\mu\nu} \mapsto g_{\mu\nu} + \delta g_{\mu\nu}$. First we compute the variation in $g^{\mu\nu}$ induced by $\delta g_{\mu\nu}$. Use

$$g^{\mu\nu} g_{\nu\beta} = \delta_\alpha^\mu \Rightarrow$$

$$\delta g^{\mu\nu} g_{\nu\beta} + g^{\mu\nu} \delta g_{\nu\beta} = 0$$

$$\Rightarrow \delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\beta} \delta g_{\rho\beta}$$
Next we compute \( \delta \sqrt{-g} \). Let \( M \) be a matrix that depends on a parameter \( \lambda \). Then we have the useful identity,

\[
\frac{d}{d\lambda} (\det M) = (\det M) \text{tr} \left( M^{-1} \frac{dM}{d\lambda} \right).
\]

Identify \( M \) with \( g_{\mu\nu}, \det M = g \), this implies

\[
\delta g = g \left( g^{\mu\nu} \delta g_{\mu\nu} \right),
\]

or

\[
\delta \sqrt{-g} = \frac{1}{2} \sqrt{-g} \left( g^{\mu\nu} \delta g_{\mu\nu} \right).
\]

Finally, we need \( \delta \Gamma \). Start by with \( \delta \Gamma_{\alpha\beta}^\nu \), the change in the L.C. \( \Gamma \) when \( g_{\mu\nu} \) goes to \( g_{\mu\nu} + \delta g_{\mu\nu} \). Being the change difference between 2 connections, this is a tensor, which we will write as \((\delta \Gamma)^{\nu}_{\alpha\beta}\) to be careful about the positions of the indices. Of course \( \Gamma_{\alpha\beta} \) itself is not a tensor.

Now we compute \( \delta \Gamma^{\mu}_{\nu\alpha\beta} \) in terms of \( \delta \Gamma \). The expression for \( \Gamma \) has the structure

\[
\Gamma = \alpha \Gamma - \alpha \Gamma + \Gamma \Gamma - \Gamma \Gamma,
\]

omitting all indices. Therefore

\[
\delta \Gamma = \alpha (\delta \Gamma) - \alpha (\delta \Gamma) + (\delta \Gamma) \Gamma + \Gamma (\delta \Gamma) - (\delta \Gamma) \Gamma - \Gamma (\delta \Gamma).
\]

We evaluate \( \delta \Gamma^{\mu}_{\nu\alpha\beta} \) at an arbitrary point of the manifold that we call \( O \), \( \delta \Gamma^{\mu}_{\nu\alpha\beta}(O) \). We use Riemann normal coordinates based at \( O \), so \( \Gamma^{\mu}_{\nu\alpha}(O) = 0 \). Then

\[
\delta \Gamma^{\mu}_{\nu\alpha\beta}(O) = (\delta \Gamma)^{\mu}_{\nu\alpha\beta}(O) - (\delta \Gamma)^{\mu}_{\nu\alpha\beta}(O),
\]

since all 4 terms in \( \Gamma - \delta \Gamma \) vanish. Since \( \delta \Gamma \) is a tensor, both terms above are ordinary derivatives of tensors, evaluated at \( O \).
But in RNC, such odd derivatives are equal to covariant derivatives (evaluated at 0). So we can replace the comma with a semicolon. Then we have a relation between two tensors,

$$\delta R^\mu_{\nu \rho} (0) = (\delta \Gamma)^{\mu}_{\nu \rho} (0) - (\delta \Gamma)^\mu_{\nu \rho} (0).$$

But since 0 was arbitrary, this is true at all points,

$$\delta R^\mu_{\nu \rho} = (\delta \Gamma)^{\mu}_{\nu \rho} - (\delta \Gamma)^\mu_{\nu \rho}.$$

And since it is a tensor eqn, it is valid in all coordinates (not only RNC).

Now by contracting, we get the variation of the Riemann tensor,

$$\delta R_{\nu \beta} = (\delta \Gamma)_{\nu \rho}^{\alpha} - (\delta \Gamma)_{\nu \rho}^{\alpha}.$$

or jiggling indices

$$\delta R_{\mu \nu} = (\delta \Gamma)^{\alpha}_{\nu \mu ; \alpha} - (\delta \Gamma)^{\alpha}_{\nu \mu ; \alpha}.$$

Finally, as for the curvature scalar, we have $R = g^{\mu \nu} R_{\mu \nu} = R_{\mu \nu}^{\mu \nu}$,

$$\delta R = \delta g^{\mu \nu} R_{\mu \nu} + g^{\mu \nu} \delta R_{\mu \nu}$$

$$= -g^{\mu \alpha} g^{\nu \beta} \delta g_{\alpha \beta} R_{\mu \nu} + g^{\mu \nu} (\delta \Gamma)^{\alpha}_{\nu \mu ; \alpha} - (\delta \Gamma)^{\alpha}_{\nu \mu ; \alpha}$$

$$\delta R = -R^{\mu \nu} \delta g_{\mu \nu} + (\delta \Gamma)^{\alpha \mu}_{\nu ; \alpha} - (\delta \Gamma)^{\alpha \mu}_{\nu ; \alpha}.$$

Thus,

$$\delta \int d^4 x \sqrt{-g} \ R = \int d^4 x \left[ \delta \sqrt{-g} \ R + \sqrt{-g} \ \delta R \right]$$

$$= \int d^4 x \sqrt{-g} \left[ \frac{1}{2} \ g^{\mu \nu} \delta g_{\mu \nu} + R^{\mu \nu} \delta g_{\mu \nu} + (\delta \Gamma)^{\alpha \mu}_{\nu ; \alpha} - (\delta \Gamma)^{\alpha \mu}_{\nu ; \alpha} \right]$$

$$= 4 \ \text{terms.}$$
The last two terms vanish on integration. For example, let

\[ x^\alpha = 8\Gamma^\alpha_{\mu\nu} \mu, \]

so the expression

\[ x^\alpha \partial_\alpha \]

(the covariant divergence of a vector) appears in the integral. This can also be written,

\[ x^\alpha ; \alpha = \frac{1}{\sqrt{-g}} (\nabla - g^\alpha_\nu x^\nu) , \alpha \]

by an identity we will prove shortly. Thus

\[ \int d^4x \sqrt{-g} \ x^\alpha ; \alpha = \int d^4x \ (\nabla - g^\alpha_\nu x^\nu) , \alpha = 0 \]

by integration by parts (X vanishes at \( \infty \)). (Or maybe \( M \) is compact.) Similarly for the 4th term. Thus,

\[ 8 \int d^4x \sqrt{-g} \ R = \int d^4x \sqrt{-g} \left[ + \frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} \right] 8 g_{\mu\nu} = 0 \]

for all \( 8 g_{\mu\nu} \Rightarrow \)

\[ + \frac{1}{2} g^{\mu\nu} R - R^{\mu\nu} = - G^{\mu\nu} = 0 \]

the vacuum Einstein equations.

Conventionally we take

\[ L_G = \frac{R}{16\pi G} \]

\( G = \) Newton's constant of gravitation, henceforth set to 1.

If a matter Lagrangian \( L_m \) is added to \( L_G \) and the overall variational principle is
$$8 \int d^4x \sqrt{-g} \left( L_g + L_M \right) = 0$$

with $L_g = R/16\pi$, then to get the right field equations,

$$G_{\mu\nu} = 8\pi T_{\mu\nu}$$

we must have

$$\frac{8}{5g_{\mu\nu}} \int d^4x \sqrt{-g} \ L_M = \frac{1}{2} T^{\mu\nu}.$$ 

Now we turn to the problem of putting spinors into curved space-time. The idea is to add the Dirac Lagrangian to the gravitational one $L_g$. In special relativity (SR), the Dirac Lagrangian is

$$L_D = \bar{\Psi} \left( i \gamma^\mu \partial_\mu - m \right) \Psi$$

in units $\hbar = c = 1$. Notation is standard, $\gamma^\mu$ are the Dirac 4x4 matrices, $\Psi$ is the Dirac 4-spinor, and $\partial_\mu$ means differentiation wrt. flat space coordinates $(t, \vec{x}) = x^\mu$.

There are two problems on putting this into GR. The first is that the usual $\gamma$ matrices are tied to inertial frames in SR, i.e. coordinates $x^\mu = (t, \vec{x})$. Rather than trying to generalize the $\gamma$ matrices to other frames, a better choice is to introduce an orthonormal vierbein $e^\mu_\nu$,

$$g_{\mu\nu} = \eta_{\mu\nu},$$

and replace $\partial_\mu$ by $e^\mu_\nu \partial_\nu$. Then we can use the standard $\gamma$ matrices of SR even in GR.

The second problem is that $\frac{\delta}{\delta \Psi} \ E_\alpha \Psi$ (or use our generalized comma notation) is not covariant, so $L_D$ is not a scalar $\Box$ in GR, as written. Obviously we must replace $E_\alpha \Psi$
with $\nabla \psi$, where $\nabla = \nabla_\alpha$ is a covariant derivative. But how do we compute covariant derivatives of spinors?

Take our clue from the covariant derivative of ordinary vectors. Begin with parallel transport of vector $y \in T_x M$ to $y' \in T_{x+\Delta x} M$,

$$
\begin{array}{c}
y \\
\Delta x \\
x+\Delta x \\
y'
\end{array}
$$

In some local chart $x^\mu$, we know that

$$
y'^\mu = (\delta^\mu_\nu - \Delta x^\x (\nabla^\mu_\nu)) y^\nu
$$

where $\nabla^\mu_\nu$ are the connection coefficients w.r.t. the chart $x^\mu$. If we transform this to an ON vierbein $e^a_\nu$, then we have

$$
y'^a = (\delta^a_\beta - \Delta x^\gamma (\nabla^a_\beta)) y^\beta,
$$

where now $\nabla^a_\beta$ is the connection coefficients w.r.t. to the vierbein. It is an equation of the same form, in spite of the fact that $\nabla$ does not transform as a tensor. Now, however, the matrix

$$
\Delta^a_\beta = (I + \Omega^a_\beta) = \delta^a_\beta - \Delta x^\gamma \Gamma^a_\gamma\beta
$$

is an infinitesimal Lorentz transformation, where the correction term

$$
\Omega^a_\beta = -\Delta x^\gamma \Gamma^a_\gamma\beta
$$

satisfies $\Omega^a_\beta = -\Omega^a_\beta$ (it is an element of the Lie algebra of $O(3,1)$.)
To parallel transport Dirac spinors from $x$ to $x+\Delta x$, say,

\[
\psi \rightarrow \psi'
\]

we may apply the Lorentz transformation $D(\Lambda)$ to $\psi$, where $\Lambda = I + \mathcal{A}$ is the infinitesimal Lorentz transformation defined above. Here $D(\Lambda)$ is the representation of the Lorentz group for Dirac spinors. Actually, it is not a representation, since it is double-valued. More about that next week.