We have seen that integrating r-forms over r-dimensional submanifolds is not general enough. For example, with \( r=1 \), we need to integrate over paths, that is, functions

\[ f: I \rightarrow M \]

\( I = [0,1] \) = Standard region \( \subset \mathbb{R} \)

If \( \omega = \omega_\mu(x) dx^\mu \) is a 1-form on \( M \) (in chart \( x^\mu \) on \( M \)), then the integral we want is

\[ \int_I \omega = \int_0^1 \omega_\mu(x(t)) \frac{dx^\mu}{dt} dt = \int_{\sigma} f^* \omega \]

where the last integral is that of a 1-form over a region of \( \mathbb{R}^r \), defined previously (Step 1 above). More generally, let us call a map

\[ \sigma: I^r \rightarrow M \]

a singular r-cube. \( I^r \subset \mathbb{R}^r \) is the r-cube, a standard region in \( \mathbb{R}^r \); the word "singular" is added to talk about the map \( \sigma \), which need not be injective, nor need \( \sigma \circ \tau^\mu \) have maximal rank. For example, in \( \sigma \) (a subset of \( M \)) need not have dimension \( r \), it may have self-intersections, etc. It need not be an r-dimensional submanifold of \( M \).

Some authors (e.g. Nakahara) prefer to use a different standard region in \( \mathbb{R}^r \), such as a simplex. Then the map is referred to as a singular simplex. There is no loss of generality in using cubes.
We now define the integral of an \( r \)-form \( \omega \in \Omega^r(M) \) over a singular \( r \)-cube. It is

\[
\int_{\sigma} \omega = \int_{I^r} \sigma^* \omega
\]

which reduces the integral to the integral of an \( r \)-form over an \( r \)-dimensional region of \( \mathbb{R}^r \). To put this in coordinates, let \( x^\mu \) be coordinates on \( M \) (\( \mu = 1, \ldots, m = \dim M, \ m \geq r \)), and let \( u^\alpha, \alpha = 1, \ldots, r \) be the standard (Euclidean) coordinates on \( \mathbb{R}^r \). Then

\[
\int_{\sigma} \omega = \int_{u^\prime} \cdots \int_{u^r} \omega_{\mu_1 \ldots \mu_r}(x(u)) \frac{\partial x^{\mu_1}}{\partial u^1} \cdots \frac{\partial x^{\mu_r}}{\partial u^r}.
\]

The most general integral is taken over linear combinations of singular \( r \)-cubes. We consider only real coefficients here. If \( \{\sigma_i^r\} \) is a set of singular \( r \)-cubes, then we define

\[
C^r = \sum_i a_i \sigma_i^r, \quad a_i \in \mathbb{R}
\]

as an \( r \)-chain. Integrals over \( r \)-chains are computed by

\[
\int_{C^r} \omega = \sum_i a_i \int_{\sigma_i^r} \omega.
\]

The set of all \( r \)-chains on \( M \) is the \( r \)-th chain group, \( C^r(M, \mathbb{R}) \) (we will drop the \( \mathbb{R} \), it being henceforth understood). The \( r \)-th chain group is a group \# in the sense that it is a vector space (an Abelian group). This is like the simplicial chain groups.
considered earlier, except now they include singular cubes, and now they are \( \infty \)-dimensional.

We now define the boundary operator \( \partial \) when acting on singular \( n \)-cubes. Once that is defined, \( \partial \) becomes defined on chains by linearity. Consider e.g. \( n = 3 \).

We have 6 faces, \( i = 1, \ldots, 6 \). Each face will be associated with a singular 2-cube. But a singular 2-cube is a map from the std 2-cube (square) in \( \mathbb{R}^2 \) to \( M \), and the faces of \( I^3 \subset \mathbb{R}^3 \) are subsets of \( \mathbb{R}^3 \), not \( \mathbb{R}^2 \). So we introduce new maps \( g_i : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) that map \( I^2 \subset \mathbb{R}^2 \) onto a face of \( I^3 \subset \mathbb{R}^3 \). Let \( \mathbf{x}' \) be coords in \( \mathbb{R}^3 \) and \( \mathbf{u}' \) be coords in \( \mathbb{R}^2 \) (or \( (x, y, z) \) and \( (u, v) \)).

The map \( g_i \) must assign the right orientation to the face, defined by saying that \( (\mathbf{n}, \mathbf{\partial u}', \mathbf{\partial v}' ) \) are positively oriented in \( \mathbb{R}^3 \), where \( \mathbf{n} \) is an outward normal to the face, and \( \mathbf{\partial u}', \mathbf{\partial v}' \) span the face. For example, in the diagram above, we map \( I^2 \) onto the face of \( I^3 \) by writing,

\[
\begin{align*}
\mathbf{u} &= z \\
\mathbf{v} &= x \\
1 &= y
\end{align*}
\]
Then
\[ \nabla : \mathbb{T}^2 \to M \]
\text{in the } i\text{-th face, a singular } 2\text{-cube. This defines } \nabla : C^r(M) \to C^{r-1}(M) \text{.}

Then we define, as in homology theory,
\[
\begin{align*}
Z_r(M) &= \{ c \in C_r(M) \mid \partial c = 0 \} = \text{r-th cycle group} = \ker d_r \\
B_r(M) &= \{ c \in C_r(M) \mid c = \partial b, \text{ some } b \in B_{r+1} \} = \text{r-th boundary group} = \text{im } d_{r+1}.
\end{align*}
\]
And we have \( \partial^2 = 0 \), as before, so \( B_r(M) \subset Z_r(M) \). And we define
\[
H_r(M) = \frac{Z_r(M)}{B_r(M)} = \text{r-th homology group.}
\]

This is the same group as before.

The properties of \( \nabla \) on chains is mirrored in the properties of \( d \) acting on forms. The terminology reflects this:
\[
\begin{align*}
\Omega^r(M) &= \{ \omega \text{ r-forms on } M \} \\
\Omega^r(M) &= \Omega^r(M) \\
\Omega^r(M) &= \Omega^r(M) \\
\text{closed forms} &\quad \to \quad Z^r(M) = \{ \omega \in \Omega^r(M) \mid d\omega = 0 \}, \text{ r-th cocycle group} = \ker d_r \\
\text{exact forms} &\quad \to \quad B^r(M) = \{ \omega \in \Omega^r(M) \mid \omega = d\beta, \text{ some } \beta \in \Omega^{r-1}(M) \}, = \text{im } d_{r-1} \text{ r-th coboundary group}
\end{align*}
\]
And because \( d^2 = 0 \), we have \( B^r(M) \subset Z^r(M) \). And we define
\[
H^r(M) = \frac{Z^r(M)}{B^r(M)} = \frac{\text{closed}}{\text{exact}} = \frac{\text{cocycles}}{\text{coboundaries}} = \text{r-th cohomology group.}
\]
To explore this association, we need Stokes' theorem, which says, if \( c \subset C^{n+1}(M) \), \( \omega \in \Omega^n(M) \),

\[
\int_c \omega = \oint_c d\omega
\]

To prove this, it suffices to consider a single singular r-cube, since chains are lin. comb's. of such things. We do example of 3-forms. Let \( \text{dim } M = m \) anything. Let \( \omega \in \Omega^2(M) \), so \( d\omega \in \Omega^3(M) \).

\[ c = \text{chain (} f: I^3 \to M \text{)}. \]

Let \( \alpha = f^*\omega \), \( \alpha \in \Omega^2(\mathbb{R}^3) \). This means \( \alpha \) has 3 nonzero components,

\[ \alpha = \alpha_x \, dy \wedge dz + \alpha_y \, dz \wedge dx + \alpha_z \, dx \wedge dy, \]

\[ d\alpha = d(f^*\omega) = f^*(d\omega) = \left( \frac{\partial \alpha_x}{\partial x} + \frac{\partial \alpha_y}{\partial y} + \frac{\partial \alpha_z}{\partial z} \right) \, dx \wedge dy \wedge dz \]

So,

\[ \int_c d\omega = \int_{I^3} d\alpha = \int_0^1 dx \int_0^1 dy \int_0^1 dz \left( \frac{\partial \alpha_x}{\partial x} + \frac{\partial \alpha_y}{\partial y} + \frac{\partial \alpha_z}{\partial z} \right) = 3 \text{ terms}. \]

Look at z-term, \[ \int \int \left[ \alpha_z(x, y, 1) - \alpha_z(x, y, 0) \right]. \]
We get 6 terms altogether for $\int dw$. Now consider $\int \omega$.

Look at the top face of the cube: let $g_i : I^2 \to \text{top face of } I^3$.

\[ x = u \]
\[ y = v \]
\[ z = 1 \]

Then $g_i^* \omega = g_i^* f^* \omega = g_i^* \alpha$. But

\[ g_i^* \alpha = \int_{I^2} \alpha_i(\xi, \eta) \, d\xi d\eta = \alpha_i(u, v) \, du dv, \]

since $dz = 0$ on top face. Thus,

\[ \int (f \circ g_i)^* \omega = \int_{I^2} g_i^* \alpha = \int_0^1 \int_0^1 \alpha_i(u, v, 1). \]

This is one of the 6 terms from the integral $\int dw$. The other 5 add up to make $\int dw$.

---

**Table:**

<table>
<thead>
<tr>
<th>Homology</th>
<th>Cohomology</th>
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<tr>
<td>$C_r(M)$</td>
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<tr>
<td>$Z_r(M)$</td>
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<tr>
<td>$B_r(M)$</td>
<td>$B^r(M)$</td>
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<tr>
<td>$H_r(M)$</td>
<td>$H^r(M)$</td>
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</table>
These spaces are dual to each other, in a certain sense. Notation, let $\omega \in \Omega^r(M)$, $c \in C_r(M)$, then write

$$\int_c \omega = (\omega, c) \in \mathbb{R}.$$ 

Thus $r$-forms are real-valued, linear operators on the space of $r$-chains, and vice versa. This suggests that maybe $\Omega^r(M)$ is the dual space to $C_r(M)$,

$$\Omega^r(M) = C^*_r(M).$$

These are co-dimensional vector spaces, so making this interpretation precise involves a big effort. We will just proceed as if it is true.

In an earlier HW problem, had vector space $V$, its dual $V^*$, a subspace $U \subseteq V$, and $X^* \subseteq V^*$, where $X^*$ is set of forms that annihilate $U$. Then we had a thm,

$$\dim U + \dim X^* = \dim V \quad (\ker X^* = U)$$

so if $U$ is big (high dimensionality), $X^*$ is small and vice versa.

(You can specify a vector subspace $U \subseteq V$ either by vectors that span it, or by forms that annihilate it.)

So what are the forms in $\Omega^r(M)$ that annihilate $Z_r(M) \subseteq C_r(M)$? (Note if they annihilate $Z_r$, they annihilate $B_r$, too). Answer: $B^r(M)$ (coboundaries, or exact forms). How to see: let $\beta \in B^r(M)$, $z \in Z_r(M)$ [$\beta = \text{exact}$, $z = \text{a cycle}$]. Then ($\beta = dy$, some $y \in \Omega^{r-1}(M)$).

$$\int_{\gamma} \beta = \int_{\gamma} \partial y = \int_{\partial z} y = 0.$$
And what are the forms that annihilate \( \mathbb{B}_r(M) \subset \mathbb{Z}_r(M) \subset \mathbb{C}_r(M) \)?

**Ans:** \( \mathbb{Z}^r(M) \) (cocycles, or closed forms). How to see: Let \( b \in \mathbb{B}_r(M) \) (a boundary, so \( b = \partial c \), some \( c \in \mathbb{C}^{r+1}(M) \)), and let \( \xi \in \mathbb{Z}_r(M) \), (a closed form, \( d\xi = 0 \)). Then

\[
\int \xi \bigg|_b - \int \xi \bigg|_c = \int d\xi = 0.
\]

Conversely, interpreting \( \mathbb{C}_r(M) \) as the operators and \( \Omega^r(M) \) as the operands, then \( \mathbb{B}_r(M) \) is the space that annihilates \( \mathbb{Z}^r(M) \), and \( \mathbb{Z}_r(M) \) annihilates \( \mathbb{B}^r(M) \).

---

What is the space dual to \( \mathbb{H}_r(M) \) (homology group)?

An element of \( \mathbb{H}_r(M) \) is \([z]\) where \( z \in \mathbb{Z}_r(M) \) is a cycle and \([z] = [z + b]\) where \( b \in \mathbb{B}_r(M) \) is a boundary. So, an operator acting on \( \mathbb{H}_r(M) \) would be one that acts on \( \mathbb{Z}_r(M) \) but annihilates boundaries, so the answer does not depend on which cycle \( z \) in \([z]\) is chosen. This means it should be a cocycle, because if \( \xi \in \mathbb{Z}^r(M) \), then

\[
(\xi, z + b) = (\xi, z) + (\xi, b).
\]

So, \( \xi \in \mathbb{Z}^r(M) \) can be associated with an element of \( \mathbb{H}_r(M)^* \). (You can think of \( (\xi, [z]) \).) However, this element of \( \mathbb{H}_r(M)^* \) is not uniquely specified by \( \xi \), because \( \xi' = \xi + \beta \), where \( \beta \in \mathbb{B}^r(M) \), \( \beta = dy \), specifies the same map: \( \mathbb{H}_r(M) \to \mathbb{R} : \)

\[
(\xi + \beta, z) = (\xi, z) + \left(\beta, z\right) \xrightarrow{d\gamma} (d\gamma, z) = 0.
\]
Thus, the element of $H_r(M)^*$ is specified by an equivalence class $[\xi] = [\xi + \rho]$, $\rho = dy$, that is, an element of $H^r(M)$. This suggests that

$$H_r(M)^* = H^r(M).$$

De Rham's Theorem asserts that this is correct, and moreover that in the case $M$ is compact, $H^r(M)$ is finite dimensional. This dimensionality is the $r$-th Betti number,

$$b_r = \dim H_r(M) = \dim H^r(M).$$

$H^r(M)$ is properly called the $r$-th de Rham cohomology group.

**Remark:** In Stokes' thm,

$$(\omega, \partial c) = (d\omega, c)$$

we can see that $d$ is the pull-back of $\partial$. That is,

$$\partial_r : C^r(M) \to C^{r-1}(M)$$

$$\partial^*_r : C^{r-1}(M)^* \to C^r(M)^*$$

$$C^{r-1}(M) \to C^r(M)$$

Thus $d_{r-1} = \partial^*_r$. Nakahara mistakenly calls this the adjoint (which requires a metric).